MAT 351: Partial Differential Equations
November 17, 2017

A **Hilbert space** is a vector space $\mathcal{H}$ over $\mathbb{C}$ (or $\mathbb{R}$) with an **inner product** $\langle u, v\rangle$ that is

- **linear** in the first slot: $\langle a_1u_1 + a_2u_2, v\rangle = a_1\langle u_1, v\rangle + a_2\langle u_2, v\rangle$ for $a_1, a_2 \in \mathbb{C}$
- **Hermitian:** $\langle u, v\rangle = \overline{\langle v, u\rangle}$
- **positive definite:** $\langle u, u\rangle \geq 0$, with equality only for $u = 0$

such that $\mathcal{H}$ is **complete** under the norm $||u|| = \big(\langle u, u\rangle\big)^{\frac{1}{2}}$, in the sense that every Cauchy sequence in $\mathcal{H}$ converges to a limit in $\mathcal{H}$. The most important examples are the finite-dimensional complex vector spaces $\mathbb{C}^m$ with inner product $\langle u, v\rangle = \sum_i u^i\bar{v}^i$, and the function space $L^2(a, b)$ with inner product $\langle u, v\rangle = \int_a^b u(x)\bar{v}(x)\,dx$.

All properties of Euclidean space $\mathbb{C}^d$ ($\mathbb{R}^d$) that involve only finitely many vectors continue to hold in Hilbert spaces; in fact, finite-dimensional subspaces of $\mathcal{H}$ are Euclidean. For example, we have

- **Schwarz inequality:** $|\langle u, v\rangle| \leq ||u||\,||v||$, and
- the **parallelogram identity:** $||u + v||^2 + |u - v||^2 = 2\big(||u||^2 + ||v||^2\big)$,

Two vectors $u, v \in \mathcal{H}$ are **orthogonal**, if $\langle u, v\rangle = 0$. In that case, we write $u \perp v$.

- **Pythagoras' identity:** If $u \perp v$, then $||u + v||^2 = ||u||^2 + ||v||^2$.

If $W \subset \mathcal{H}$ is a finite-dimensional subspace, then the orthogonal projection onto W is given by

$$P_W u = \sum_{n=1}^{d} \langle u, w_n\rangle w_n\,,$$

where $w_1, \dots, w_n$ is any orthonormal basis of W. With this definition, $P_W u$ is the element of W that is closest to u, and thus $(P_w)^2u = P_W u$. Moreover, $u - P_W u$ is orthogonal to W. An important consequence of Pythagoras' theorem is

- **Bessel's inequality:** If $(w_n)_{n\geq 1}$ is a (finite or infinite) orthonormal sequence in $\mathcal{H}$, then

  $$||u||^2 \geq \sum_{n\geq 1} |\langle u, w_n\rangle|^2$$

  for every $u \in \mathcal{H}$. In particular, the series always converges, which in turn implies that the series $\sum_{n\geq 1}\langle u, w_n\rangle w_n$ converges in $\mathcal{H}$.

Theorem *Let $(w_n)_{n\geq 1}$ be an infinite orthonormal sequence in $\mathcal{H}$. The following are equivalent:*

1. *Finite linear combinations $\sum_{n=1}^{N} b_n w_n$ are* **dense** *in $\mathcal{H}$;*
2. **Completeness:** *If $\langle u, w_n\rangle = 0$ for all n then $u = 0$;*
3. **Parseval's identity:** *For each $u \in \mathcal{H}$, $||u||^2 = \sum_n |\langle u, w_n\rangle|^2$;*
4. *(w_n) is an* **orthonormal basis:** *Every $u \in \mathcal{H}$ can be represented as a convergent series*

$$u = \sum_{n=1}^{\infty} a_n w_n$$

for suitable coefficients $(a_n)_{n\geq 1}$.

We will prove in a few weeks that $\sqrt{\frac{2}{\pi}}(\sin nx)_{n\geq 1}$ and $\sqrt{\frac{2}{\pi}}(\cos nx)_{n\geq 0}$ are orthonormal bases for $L^2(0,\pi)$, and that $\frac{1}{\sqrt{2\pi}}(e^{inx})_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(-\pi,\pi)$. Thus each of the classical Fourier series of an L^2-function f converges in L^2 to f. (We say that the Fourier series **represents** the function.) A more subtle question is under what conditions a Fourier series converges pointwise or even uniformly to f. There are examples of continuous 2π-periodic functions whose Fourier series diverges for every x!

Read: Chapter 5.1-5.4

Please remember: Our first midterm test takes place Friday November 24, in tutorial/class. The test covers lectures and tutorials up to November 17, and Assignments 1-9.

For discussion and practice:

1. Let f be a smooth 2π-periodic function with $\int_{-\pi}^{\pi} f(x)\,dx = 0$. Use the Fourier series representation and Parseval's identity to show that $||f|| \leq ||f'||$.