MAT 351: Partial Differential Equations
Assignment 6 — October 20, 2017

We turn to inhomogeneous equations diffusion and wave equations. We formally write an inhomogeneous PDE as

$$u_t = Lu + f(\cdot, t)\,, \quad u(\cdot, 0) = \phi\,, \tag{1}$$

where L is a linear differential operator that involves only the x variable, f is the inhomogeneity or **source term**, and ϕ is the initial condition.

- To ground ourselves, let us first consider a linear **ODE**

  $$\frac{dy}{dt} = Ay + f(t) \quad y(0) = y_0\,, \tag{2}$$

  where A is a $n \times n$ matrix f is a given function, $y_0 \in \mathbb{R}^n$ a given vector, and the unknown function $y(t)$ takes values in $\mathbb{R}^n$. **Duhamel's principle** says that the solution is given by

  $$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)} f(s)\, ds\,.$$

  Here, e^{At} denotes the solution operator for the homogeneous equation $\frac{d}{dt}y = Ay$. By definition, $y(t) = e^{At}y_0$ solves the homogeneous equation with initial value $y(0) = y_0$.

  There are many equivalent ways to define and compute the matrix-valued function e^{At}, by its power series, by diagonalizing A, or by special techniques such as contour integrals. Duhamel's formula is proved by the method of **variation of constants**.

- For the linear **transport equation**

  $$u_t + bu_x = f(x,t)\,, \quad u(x,0) = \phi(x)\,, \tag{3}$$

  we write $L = -b\partial_x$. The solution of the homogeneous problem is $e^{tL}\phi(x) = \phi(x - bt)$. By Duhamel's principle,

  $$u(x,t) = \phi(x - bt) + \int_0^t f(x - b(t-s), s)\, ds\,.$$

- Let us now try to solve the **heat equation with sources**

  $$u_t = ku_{xx} + f(x,t)\,, \quad u(x,0) = \phi(x)\,. \tag{4}$$

  Here, $L = k\partial_x^2$. Denote by e^{tL} the solution operator for the heat equation, given by

  $$\left(e^{tL}\phi\right)(x) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\, dy\,,$$

  where $S(x,t) = (4\pi kt)^{-1/2}e^{-x^2/(4t)}$ is the fundamental solution. By Duhamel's principle, the solution of the inhomogeneous equation is given by

  $$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\,\phi(y)\, dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\, f(y,s)\, dyds\,.$$

- Consider finally the initial-value problem for the wave equation with sources,

$$u_{tt} = c^2 u_{xx} + f(x,t)\,, \quad u(x,u) = \phi(x), u_t(x,0) = \psi(x)\,. \tag{5}$$

  Set $v = u_t$. Then the wave equation becomes the system

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ c^2\partial_x^2 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f(x,t) \end{pmatrix} =: L \begin{pmatrix} u \\ v \end{pmatrix} + F(x,t)\,.$$

  We then use D'Alembert's formula to write the solution operator $e^{tL}(\phi, \psi)$ for the homogeneous equation. Plugging this into Duhamel's formula gives

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct) + \phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dyds\,.$$

Duhamel's formula is a recipe how to construct solutions of inhomogeneous evolution equations. Our heuristic derivation does not constitute a proof. But once we guess the correct formulas, we can check by direct computation that they indeed satisfies Eqs. (3), (4), and (5).

Read: Chapter 3 of Strauss.

Hand-in (due Friday, October 27):

(H1) Let D be a connected open bounded set with smooth boundary in $\mathbb{R}^n$, let ∂D be its boundary, and $\overline{D}$ its closure. Let u be a solution of **Laplace's equation** $\Delta u = 0$ on D that is continuous on $\overline{D}$.

  (a) Prove that u satisfies the **maximum principle**: $\sup\limits_{x\in D} u(x) = \max\limits_{x\in\partial D} u(x)$.

  (b) Why is it necessary to assume that D is bounded?

  (c) Conclude that **Poisson's equation** $\Delta u = f(x)$ on D, with boundary conditions $u(x) = g(x)$ for $x \in \partial D$ can have at most one solution.

(H2) (a) Use the method of reflections to write the solution of the boundary-value problem

$$\begin{aligned} &u_t = u_{xx}\,, && 0 < x < 1, t > 0\,, \\ &u_x(0,t) = u_x(1,t) = 0\,, && t > 0\,, u(x,0) = \phi(x) \end{aligned}$$

  as a series. Here, ϕ is a continuously differentiable function with $\phi'(0) = \phi'(1) = 0$.

  (b) Does the series converge? In what sense?

For discussion and practice:

- Let Φ be a nonnegative real-valued smooth function on $\mathbb{R}$ with $\Phi(0) = 0$, and define

$$E(u) = \int_{\mathbb{R}} \Phi(u(x))\,dx\,.$$

  Under what conditions on Φ does $E(u(\cdot,t))$ decrease in time for *every* solution of the heat equation $u_t = ku_{xx}$? (Assume that u is smooth and $E(u(\cdot,t))$ is finite for all $t \geq 0$.)