Stat 2211 Markov chain practice problems

Do not hand in

Problem 1. Let X be a reversible Markov chain with period 2, started from some distribution μ_0. Show that X_{2n} converges in distribution to some limit that does not depend on μ_0.

Problem 2. Show that biased random walk on the integers is transient.

Problem 3. Consider a finite irreducible Markov chain, and let T_x be the return time to x of the chain started from x. Show that $\mathbf{E}T_x < \infty$. Show also that there exists constants $c > 0, a < 1$ so that $P(T_x > t) \leq ca^t$ for all t. Show that $\mu(x) = 1/\mathbf{E}(T_x)$ is a the unique stationary distribution.

Problem 4. Let X be the following random walk on the d dimensional hypercube $\{0,1\}^d$. Pick one of the d coordinates at random and replace it by a value picked uniformly at random from $\{0,1\}$. Show that the mixing time is at most $cn \log n$. (Hint: relate this to the coupon collector problem).

Problem 5. Let X be a Markov chain on the two-point space $\{0,1\}$ that at every step stays put with probability p and moves with probability $1-p$. Let $X_0 = 0$. Give an explicit formula in terms of p and n for the total variation distance of X_n from uniform distribution.

Problem 6. Consider a recurrent Markov chain with kernel K started at some vertex v. Let $T_0 = 0$ and let T_i be the ith return time to v. Let $C_n = (X_{T_{n-1}}, \ldots, X_{T_n - 1})$ be a random finite sequence for each n. This is called the nth excursion from v. Show that the C_n are independent and have the same distribution.

Problem 7. Consider the C_n from the previous problem, and reverse the order of its elements to get $\bar{C}_n$. Now concatenate $(v), \bar{C}_1, \bar{C}_2, \ldots$ to get a random infinite sequence of states of the Markov chain with kernel K. Show that this random sequence is a sample from the time-reversal of the Markov chain with kernel K with respect to the stationary measure.

Problem 8. Let G be the complete graph on 4 vertices with no loops. Compute the eigenvalues for the random walk on G, and find an orthonormal basis of eigenvectors. Specify the distribution of X_n for the walk started at a fixed vertex v.

Problem 9. Show that an irreducible Markov chain has period divisible by m if and only if the states can be divided into m groups $S_0, \ldots S_{m-1}$ so that $K(x,y) = 0$ unless $x \in S_i$ and $y \in S_{i+1}$ for some i (where $i+1$ is understood modulo m).