

Airy Phase Functions

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Abstract

It is well known that phase function methods allow for the numerical solution of a large class of oscillatory second order linear ordinary differential equations in time independent of frequency. Unfortunately, these methods break down in the commonly-occurring case in which the equation has turning points. We resolve this difficulty by introducing a generalized phase function method for second order linear ordinary differential equations with turning points. More explicitly, we prove the existence of a slowly-varying “Airy phase function” which efficiently represents the solutions of such an equation, and describe a numerical algorithm for constructing it. The running time of our algorithm is independent of the magnitude of the logarithmic derivatives of the equation’s solutions, which is a measure of their rate of variation that generalizes the notion of frequency to functions which are rapidly varying but not necessarily oscillatory. Once the Airy phase function has been constructed, any reasonable initial or boundary value problem can be readily solved and, unlike step methods that only generate the values of a rapidly-varying solution at the nodes of a sparse discretization grid that is insufficient for interpolation, the output of our scheme allows for the evaluation of the solution at any point in the equation’s domain. We rigorously justify our approach by proving not only the existence of slowly-varying Airy phase functions, but also the convergence of our numerical method. Moreover, we present the results of extensive numerical experiments demonstrating the efficacy of our algorithm.

Keywords: oscillatory problems, fast algorithms, ordinary differential equations

1. Introduction

Second order linear ordinary differential equations frequently arise in numerical and scientific computations. Many of the equations which are encountered are either of the form

$$y''(t) + \omega^2 q(t, \omega) y(t) = 0, \quad a < t < b, \quad (1)$$

with ω a real-valued parameter and q a smooth function such that q and its derivatives with respect to t are bounded independent of ω , or they can be easily put into this form. Several widely-used families of special functions, such as the Jacobi polynomials and the spheroidal wave functions, satisfy equations of this type. They also arise in computations related to plasma physics [8, 6], Hamiltonian dynamics [13] and cosmology [11, 1], to name just a few representative applications.

For the sake of notational simplicity, we will generally suppress the dependence of q on ω . This

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causes no harm because of our assumption that q and its derivatives with respect to t of all orders are bounded independent of ω . Moreover, we will focus on the commonly-occurring case in which q has a single simple turning point in its domain $[a, b]$. Without loss of generality, we will suppose that $a < 0 < b$, and that $q(t) \sim t$ as $t \rightarrow 0$. In particular, q is strictly positive on $(0, b)$ and strictly negative on $(a, 0)$. It is well known that, under these conditions, there exists a basis $\{y_1, y_2\}$ in the space of solutions of (1) which can be asymptotically approximated on various portions of the interval $[a, b]$ as follows:

(a) In any compact subinterval of $(0, b]$, the estimates

$$y_1(t) = \frac{\cos(\alpha_0(t))}{\sqrt{\alpha'_0(t)}} \left(1 + \mathcal{O}\left(\frac{1}{\omega}\right)\right) \quad \text{and} \quad y_2(t) = \frac{\sin(\alpha_0(t))}{\sqrt{\alpha'_0(t)}} \left(1 + \mathcal{O}\left(\frac{1}{\omega}\right)\right) \quad (2)$$

hold uniformly with respect to t as $\omega \rightarrow \infty$, where α_0 is given by

$$\alpha_0(t) = \frac{\pi}{4} + \omega \int_0^t \sqrt{q(s)} ds. \quad (3)$$

(b) In any compact interval contained in $[a, 0)$, the estimates

$$y_1(t) = \frac{\exp(-\beta_0(t))}{\sqrt{\beta'_0(t)}} \left(1 + \mathcal{O}\left(\frac{1}{\omega}\right)\right) \quad \text{and} \quad y_2(t) = \frac{\exp(\beta_0(t))}{2\sqrt{\beta'_0(t)}} \left(1 + \mathcal{O}\left(\frac{1}{\omega}\right)\right) \quad (4)$$

hold uniformly with respect to t as $\omega \rightarrow \infty$, where β_0 is defined via

$$\beta_0(t) = \omega \int_0^t \sqrt{-q(s)} ds. \quad (5)$$

(c) In any compact subinterval of $[a, b]$, we have

$$y_1(t) = \frac{\text{Bi}(\gamma_0(t))}{\sqrt{\gamma'_0(t)}} \left(1 + \mathcal{O}\left(\frac{1}{\omega}\right)\right) \quad \text{and} \quad y_2(t) = \frac{\text{Ai}(\gamma_0(t))}{\sqrt{\gamma'_0(t)}} \left(1 + \mathcal{O}\left(\frac{1}{\omega}\right)\right) \quad (6)$$

uniformly with respect to t as $\omega \rightarrow \infty$, where γ_0 is defined by

$$\gamma_0(t) = \begin{cases} \left(\frac{3}{2}\omega \int_0^t \sqrt{q(s)} ds\right)^{\frac{2}{3}} & \text{if } t \geq 0 \\ -\left(-\frac{3}{2}\omega \int_0^t \sqrt{-q(s)} ds\right)^{\frac{2}{3}} & \text{if } t < 0 \end{cases} \quad (7)$$

and Ai and Bi refer to the solutions of Airy's differential equation

$$y''(t) + ty(t) = 0 \quad (8)$$

such that

$$\text{Ai}(0) = \frac{\sqrt{\pi}}{3^{\frac{2}{3}}\Gamma\left(\frac{2}{3}\right)}, \quad \text{Ai}'(0) = \frac{\sqrt{\pi}}{3^{\frac{1}{3}}\Gamma\left(\frac{1}{3}\right)}, \quad \text{Bi}(0) = \frac{\sqrt{\pi}}{3^{\frac{1}{6}}\Gamma\left(\frac{2}{3}\right)} \quad \text{and} \quad \text{Bi}'(0) = \frac{-3^{\frac{1}{6}}\sqrt{\pi}}{\Gamma\left(\frac{1}{3}\right)}. \quad (9)$$

We note that our definitions of Ai and Bi are nonstandard. It is more common for authors to work with the functions

$$\frac{1}{\sqrt{\pi}} \text{Ai}(-t) \quad \text{and} \quad \frac{1}{\sqrt{\pi}} \text{Bi}(-t), \quad (10)$$

which are solutions of

$$y''(t) - ty(t) = 0. \quad (11)$$

These are the definitions of the Airy functions which are used in [7] and [12], for example. However, our convention is more suitable when treating equations of the form (1) for which $q(t) \sim t$ as $t \rightarrow 0$, and our choice of normalization ensures that the Wronskian of $\{\text{Ai}(t), \text{Bi}(t)\}$, and hence also of the pair of asymptotic approximations appearing in (6), is 1.

It follows from (2), (4) and (6) that the solutions of an equation of the form (1) are rapidly varying when ω is large. Indeed, if $f(t) = y_1(t) + iy_2(t)$ and j is a positive integer, then

$$\sup_{a \leq t \leq b} \left| \frac{f^{(j)}(t)}{f(t)} \right| = \mathcal{O}(\omega^j) \text{ as } \omega \rightarrow \infty. \quad (12)$$

One obvious consequence is that the cost of representing the solutions of (1) over the interval $[a, b]$ via polynomial or piecewise polynomial expansions grows linearly with ω . Since standard solvers for ordinary differential equations use such expansions to represent solutions either explicitly or implicitly, their running times also increase linearly with ω .

In contrast to the rapidly-varying solutions of (1), α_0 , β_0 and γ_0 are slowly varying in the sense that whenever f is equal to one of these functions, $[c, d]$ is a compact interval in the domain of definition of f and j is a positive integer, the quantity

$$\sup_{c \leq t \leq d} \left| \frac{f^{(j)}(t)}{f(t)} \right| \quad (13)$$

is bounded independent of ω . It follows that α_0 , β_0 and γ_0 can be represented via polynomial expansions at a cost which is independent of ω . In particular, initial and boundary value problems for (1) can be solved with $\mathcal{O}(\omega^{-1})$ relative accuracy in time independent of ω by forming polynomial expansions of α_0 , β_0 and γ_0 and making use of the approximations (2), (4) and (6).

It is well known that there exist higher order generalizations α_M and β_M of α_0 and β_0 . Indeed, for each nonnegative integer M and compact subinterval $[c, d]$ of $(0, b]$, there exist smooth functions $\alpha : [c, d] \rightarrow \mathbb{R}$ and $\alpha_M : [c, d] \rightarrow \mathbb{R}$ such that α_M is slowly varying in the aforementioned sense,

$$\left\{ \frac{\cos(\alpha(t))}{\sqrt{\alpha'(t)}}, \frac{\sin(\alpha(t))}{\sqrt{\alpha'(t)}} \right\} \quad (14)$$

is a basis in the space of solutions of (1) given on the interval $[c, d]$ and

$$\alpha(t) = \alpha_M(t) \left(1 + \mathcal{O}\left(\frac{1}{\omega^{2(M+1)}}\right) \right) \text{ as } \omega \rightarrow \infty. \quad (15)$$

Similarly, for each compact subinterval $[c, d]$ of $[a, 0)$ and nonnegative integer M , there exist smooth functions $\beta : [c, d] \rightarrow \mathbb{R}$ and $\beta_M : [c, d] \rightarrow \mathbb{R}$ such that β_M is slowly varying,

$$\left\{ \frac{\exp(-\beta(t))}{\sqrt{\beta'(t)}}, \frac{\exp(\beta(t))}{2\sqrt{\beta'(t)}} \right\} \quad (16)$$

is a basis in the space of solutions of (1) given on the interval $[c, d]$ and

$$\beta(t) = \beta_M(t) \left(1 + \mathcal{O}\left(\frac{1}{\omega^{2(M+1)}}\right) \right) \text{ as } \omega \rightarrow \infty. \quad (17)$$

These estimates are often derived by analyzing the Riccati equation

$$r'(t) + (r(t))^2 + \omega^2 q(t) = 0 \quad (18)$$

satisfied by the logarithmic derivatives of the solutions of (1); see, for instance, [15]. We will refer to any α such that (14) is a basis in the space of solutions of (1) on some subinterval of $[a, b]$ as a trigonometric phase function for (1), and we call any β such that (16) is a basis in the space of solutions of (1) on some subinterval of $[a, b]$ an exponential phase function for (1).

While α_0 and β_0 admit simple expressions that are easy to evaluate numerically, the same is not true of α_M and β_M when M is large. These functions are given by complicated expressions that involve repeated integration and differentiation, with the consequence that neither symbolic nor numerical calculations allow for their effective evaluation. Several recently introduced methods that apply to equations of the form (1) for which q is strictly positive (i.e., in the oscillatory regime) overcome this difficulty by numerically solving various nonlinear ordinary differential equations to calculate a trigonometric phase function directly rather than constructing one of the asymptotic approximations α_M . The first such scheme, that of [3], considers the nonlinear differential equation

$$\omega^2 q(t) - (\alpha'(t))^2 + \frac{3}{4} \left(\frac{\alpha''(t)}{\alpha'(t)} \right)^2 - \frac{1}{2} \frac{\alpha'''(t)}{\alpha'(t)} = 0, \quad (19)$$

which we refer to as Kummer's equation in light of the article [10]. The solutions of (19) comprise the trigonometric phase functions for (1), and the method of [3] exploits the fact that (15) implies the existence of a solution of (19) that can be regarded as slowly varying for the purposes of numerical computation. Indeed, by choosing M to be large, we see that there exists a trigonometric phase function which agrees to machine precision accuracy with a slowly-varying function, even for modest values of ω . Of course, since almost all solutions of Kummer's equation are rapidly varying, some mechanism must be used to select this slowly-varying solution. The algorithm of [3] employs a "windowing scheme" that entails smoothly deforming the coefficient q into a constant over a portion of the solution domain.

The paper [15] introduces a faster and simpler algorithm which appears to be the current state of the art for solving equations of the form (1) in the oscillatory case. It operates by numerically constructing a slowly-varying solution of the Riccati equation (18). Once this has been done, a slowly-varying trigonometric phase function α is obtained via the formula

$$r(t) = i\alpha'(t) - \frac{1}{2} \frac{\alpha''(t)}{\alpha'(t)} \quad (20)$$

that connects the solutions of (18) to those of (19). To identify the desired solution, (18) is discretized over a small interval $[a_0, b_0]$ in the solution domain via a Chebyshev spectral method and Newton's method is applied to the resulting system of nonlinear algebraic equations. The discretization grid is chosen to be dense enough to represent the desired slowly-varying solution, but sparse enough that it does not discretize the rapidly-varying ones. A proof that the Newton iterates converge to a unique vector representing the desired solution of the Riccati equation provided that ω is sufficiently large is given in [15]. Once the values of this solution are known on the small interval $[a_0, b_0]$, an initial value problem for (18) is solved in order to extend it to $[a, b]$, and the desired slowly-varying trigonometric phase function is constructed using (20).

Both of the phase function methods of [3] and [15] run in time independent of ω , allow for the solution of any reasonable initial or boundary value problem for (1) and achieve accuracy on the order of the condition number of the problem being solved. Moreover, unlike step methods which output

the values of a rapidly-varying solution of (1) on a sparse grid of discretization nodes that does not suffice for interpolation, the phase functions produced by [3] and [15] allow for the evaluation of solutions anywhere in the interval $[a, b]$ at a cost independent of ω .

While it is shown in [4] that trigonometric phase functions can represent the solutions of an equation of the form (1) near a turning point, they exhibit complicated behavior there, with the consequence that the cost to calculate trigonometric phase functions for second order linear ordinary differential equations with turning points is high and typically increases with ω . Here, we address this difficulty by introducing a generalized phase function method adapted to equations of the form (1) with turning points.

We first show that, under our assumptions on q and for any nonnegative integer M , there exist a smooth function $\gamma : [a, b] \rightarrow \mathbb{R}$ and a smooth slowly-varying function $\gamma_M : [a, b] \rightarrow \mathbb{R}$ such that

$$\left\{ \frac{\text{Bi}(\gamma(t))}{\sqrt{\gamma'(t)}}, \frac{\text{Ai}(\gamma(t))}{\sqrt{\gamma'(t)}} \right\} \quad (21)$$

is a basis in the space of solutions of (1) and

$$\gamma(t) = \gamma_M(t) \left(1 + \mathcal{O}\left(\frac{1}{\omega^{2(M+1)}}\right) \right) \text{ as } \omega \rightarrow \infty. \quad (22)$$

We will call any function γ such that (21) is a basis in the space of solutions of (1) an Airy phase function for (1). While the estimates (15) and (17) follow easily from results which can be found in many sources, (22) appears to be novel. The standard asymptotic approximations of the solutions of (1), which are discussed in Chapter 11 of [12] and Chapter 2 of [7], among many other sources, are of the different form

$$z \frac{\left(\omega^{\frac{2}{3}}\xi(t)\right)}{\sqrt{\xi'(t)}} \sum_{j=0}^M \frac{A_j(\xi(t))}{\omega^j} + \frac{z' \left(\omega^{\frac{2}{3}}\xi(t)\right)}{\omega^{\frac{4}{3}}\sqrt{\xi'(t)}} \sum_{j=0}^M \frac{B_j(\xi(t))}{\omega^j}, \quad (23)$$

where $\xi(t) = \omega^{-\frac{2}{3}}\gamma_0(t)$ and z is taken to be Ai or Bi. We note that because (22) holds on any compact interval in the solution domain of (1), it can be combined with standard asymptotic estimates for the Airy functions to obtain (15) and (17). However, because (15) and (17) do not hold in a neighborhood of the turning point, they cannot be used to derive (22).

Airy phase functions satisfy the nonlinear differential equation

$$\omega^2 q(t) - \gamma(t)(\gamma'(t))^2 + \frac{3}{4} \left(\frac{\gamma''(t)}{\gamma'(t)} \right)^2 - \frac{1}{2} \frac{\gamma'''(t)}{\gamma'(t)} = 0, \quad (24)$$

which we will refer to as the Airy-Kummer equation, and we go on to prove that they can be computed numerically by applying a method analogous to the algorithm of [15] to (24). That is, we show that if (24) is discretized over a small interval $[-a_0, a_0]$ via a Chebyshev spectral method and Newton's method is applied to the resulting system of nonlinear algebraic equations with the first-order approximate γ_0 used to form an initial guess, the iterates converge to a vector representing the desired slowly-varying Airy phase function γ , provided a_0 is sufficiently small and ω is sufficiently large. We use γ_0 as an initial guess because, unlike the higher order approximates γ_M , it can be readily calculated. The analysis presented here to establish the estimate (22) and prove the convergence of the Newton iterations for (24) is significantly more involved than that of [15] owing to the much more complicated structure of the Airy-Kummer equation (24) vis-à-vis the Riccati

equation (18).

As a last step in our algorithm, the function γ is extended to the entire solution domain $[a, b]$ of (1) by solving an initial value problem for (24). Here, we use an adaptive Chebyshev spectral method, and the result is a collection of piecewise Chebyshev expansions representing the Airy phase function γ and its first few derivatives. Once these expansions have been obtained, any reasonable initial or boundary value problem for (1) can be easily solved. The running time of our algorithm is independent of ω and the obtained solution can be readily evaluated at any point in $[a, b]$ at a cost which is independent of ω .

The scheme of this paper can be combined with the phase function method of [15] by computing trigonometric or exponential phase functions in some regions and Airy phase functions in others, and this is sometimes more efficient than using these algorithms separately. Moreover, our method generalizes in a straightforward way to equations with multiple simple turning points: one simply constructs multiple Airy phase functions, one for each turning point, using our method. The situation regarding higher order turning points, however, is more complicated. While numerical experiments show that a straightforward generalization of our numerical algorithm works well in the setting of equations of the form (1) in which $q(t) \sim t^\sigma$ as $t \rightarrow 0$ with $\sigma > -2$, the analysis presented here only applies when σ is $-1, 0$ or 1 . In this paper, we will focus on equations with simple turning points and we leave a treatment of other cases for future work.

The remainder of this paper is structured as follows. Section 2 discusses the necessary mathematical and numerical preliminaries, and introduces the notation used in the rest of the paper. Our proof of the existence of slowly-varying Airy phase functions is given in Section 3, while Section 4 gives a proof that Newton's method converges when it is applied to a spectral discretization of (24) on a small interval containing the turning point. Section 5 details our algorithm for the numerical computation of Airy phase functions, and we present the results of numerical experiments conducted to assess its effectiveness in Section 6. We close with a few brief remarks regarding this work and future directions for research in Section 7.

2. Preliminaries

2.1. Notation and conventions

We use $\partial_j K(t, s)$ for the partial derivative of a multivariate function $K : \mathbb{R}^n \rightarrow \mathbb{R}$ with respect to its j^{th} argument. We denote the Fréchet derivative of a map $F : X \rightarrow Y$ between Banach spaces at the point x by $D_x F$. The space of bounded linear functions $X \rightarrow Y$ is $L(X, Y)$, and we say a function $F : \Omega \subset X \rightarrow Y$ given on an open subset Ω in X is continuously differentiable provided the map $\Omega \rightarrow L(X, Y)$ given by $x \mapsto D_x F$ is continuous.

We use $\|f\|_\infty$ for the uniform norm of the function f over its domain of definition. We let $C^1([c, d])$ be the Banach space of functions $f : [c, d] \rightarrow \mathbb{R}$ which have uniformly continuous derivatives. When working with $C^1([c, d])$, we will use the nonstandard norm

$$\|f\|_\omega = \omega^2 \|f\|_\infty + \omega \|f'\|_\infty, \quad (25)$$

which is defined for any positive value of ω . We will use bold symbols for elements of the Euclidean space \mathbb{R}^k , and denote the $l^\infty(\mathbb{R}^k)$ norm of a vector \mathbf{v} by $\|\mathbf{v}\|_\infty$. We use $\text{diag}(\mathbf{x})$ for the $k \times k$ diagonal matrix whose diagonal entries are the elements of the vector $\mathbf{x} \in \mathbb{R}^k$. The Hadamard or pointwise product of the vectors \mathbf{x} and \mathbf{y} is denoted $\mathbf{x} \circ \mathbf{y}$. Moreover, if $\mathbf{v} = (v_1 \ v_2 \ \cdots \ v_k)^\top$,

then, for each integer j , we use $\mathbf{v}^{\circ j}$ for the vector

$$\left(v_1^j \quad v_2^j \quad \cdots \quad v_k^j \right)^\top. \quad (26)$$

We denote the Chebyshev polynomial of degree j by T_j , and use $t_{1,k}^{\text{cheb}}, t_{2,k}^{\text{cheb}}, \dots, t_{k,k}^{\text{cheb}}$ for the nodes of the k -point Chebyshev extremal grid on $[-1, 1]$, which are given by the formula

$$t_{i,k}^{\text{cheb}} = \cos \left(\pi \frac{k-i}{k-1} \right). \quad (27)$$

The $k \times k$ identity matrix is \mathcal{I}_k , and we use \mathcal{D}_k for the $k \times k$ spectral differentiation matrix which takes the vector

$$\left(p \left(t_{1,k}^{\text{cheb}} \right) \quad p \left(t_{2,k}^{\text{cheb}} \right) \quad \cdots \quad p \left(t_{k,k}^{\text{cheb}} \right) \right)^\top \quad (28)$$

of values of a polynomial p of degree less than k at the Chebyshev nodes to the vector

$$\left(p' \left(t_{1,k}^{\text{cheb}} \right) \quad p' \left(t_{2,k}^{\text{cheb}} \right) \quad \cdots \quad p' \left(t_{k,k}^{\text{cheb}} \right) \right)^\top \quad (29)$$

of the values of its derivatives at the same nodes.

2.2. The Newton-Kantorovich theorem

In [9], Kantorovich generalized Newton's method to the case of maps between Banach spaces and gave conditions for its convergence. Here, we state a simplified version of the Newton-Kantorovich theorem that can be found as Theorem 7.7-4 in Section 7.7 of [5]. Note that we have corrected a minor typo in condition (5) of the theorem.

Theorem 1. *Suppose that Ω is an open subset of the Banach space X , that Y is a Banach space and that $F : \Omega \subset X \rightarrow Y$ is continuously differentiable. Suppose also that there exist a point $x_0 \in \Omega$ and constants λ and η such that*

1. $D_{x_0}F$ admits an inverse $(D_{x_0}F)^{-1} \in L(Y, X)$,
2. $B_\eta(x_0) \subset \Omega$,
3. $0 < \lambda < \frac{\eta}{2}$,
4. $\left\| (D_{x_0}F)^{-1} F(x_0) \right\| \leq \lambda$ and
5. $\left\| (D_{x_0}F)^{-1} (D_xF - D_yF) \right\| \leq \frac{1}{\eta} \|x - y\|$ for all $x, y \in B_\eta(x_0)$.

Then, D_xF has a bounded inverse $D_xF^{-1} \in L(Y, X)$ for each $x \in B_\eta(x_0)$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n - (D_{x_n}F)^{-1} F(x_n) \quad (30)$$

is contained in the open ball

$$B_{\eta^-}(x_0), \quad \text{where} \quad \eta^- = \eta \left(1 - \sqrt{1 - \frac{2\lambda}{\eta}} \right) \leq \eta \quad (31)$$

and $\{x_n\}$ converges to a zero x^* of F . Moreover, x^* is the only zero of F in the ball $B_\eta(x_0)$ and, for each $n \geq 0$, we have

$$\|x_n - x^*\| \leq \frac{\eta}{2^n} \left(\frac{\eta^-}{\eta} \right)^{2^n}. \quad (32)$$

2.3. A Green's function for an auxiliary differential equation related to the Airy functions

In Section 3, we use a Green's function $G(t, s)$ for the differential equation

$$z'''(t) + 4tz'(t) + 2z(t) = 0 \quad (33)$$

in order to form an integral operator to which we then apply a contraction mapping argument. It is critical that $G(t, s)$ and its first order partial derivatives $\partial_1 G(t, s)$ and $\partial_2 G(t, s)$ be bounded for all $(t, s) \in \mathbb{R} \times \mathbb{R}$. In addition, it will be necessary to choose a Green's function which enforces the condition $z(0) = 0$.

Fortunately, because the solutions of (33) are explicitly known, it is straightforward to construct a Green's function with the desired properties. Indeed,

$$\varphi_1(t) = 2^{-\frac{1}{3}} \text{Ai}^2(t), \quad \varphi_2(t) = 2^{-\frac{1}{3}} \text{Bi}^2(t) \quad \text{and} \quad \varphi_3(t) = 2^{-\frac{1}{3}} \text{Ai}(t) \text{Bi}(t) \quad (34)$$

is a triple of solutions of (33) whose Wronskian is 1, and the method of variations of parameters can be used to show that

$$G_0(t, s) = \begin{cases} -\varphi_2(t) (\varphi_1(s)\varphi_3'(s) - \varphi_1'(s)\varphi_3(s)) + \varphi_3(t) (\varphi_1(s)\varphi_2'(s) - \varphi_1'(s)\varphi_2(s)) & \text{if } s \leq t \\ -\varphi_1(t) (\varphi_2(s)\varphi_3'(s) - \varphi_2'(s)\varphi_3(s)) & \text{if } s \geq t \end{cases} \quad (35)$$

is a Green's function for (33). The well-known asymptotic estimates

$$\text{Ai}(t) \sim t^{-\frac{1}{4}} \cos\left(\frac{\pi}{4} - \frac{2}{3}t^{\frac{3}{2}}\right), \quad \text{Bi}(t) \sim t^{-\frac{1}{4}} \sin\left(\frac{\pi}{4} - \frac{2}{3}t^{\frac{3}{2}}\right) \quad \text{as } t \rightarrow \infty \quad (36)$$

and

$$\text{Ai}(t) \sim \frac{1}{2}t^{-\frac{1}{4}} \exp\left(-\frac{2}{3}(-t)^{\frac{3}{2}}\right), \quad \text{Bi}(t) \sim t^{-\frac{1}{4}} \exp\left(\frac{2}{3}(-t)^{\frac{3}{2}}\right) \quad \text{as } t \rightarrow -\infty \quad (37)$$

imply that $G_0(t, s)$ is bounded on $\mathbb{R} \times \mathbb{R}$. Moreover, the estimates (36) and (37) can be differentiated in order to see that the partial derivatives $\partial_1 G_0(t, s)$ and $\partial_2 G_0(t, s)$ are also bounded for all $(t, s) \in \mathbb{R} \times \mathbb{R}$. Since $\text{Ai}(0) \neq 0$, we can now define

$$G(t, s) = G_0(t, s) - \frac{\text{Ai}^2(t)}{\text{Ai}^2(0)} G_0(0, s) \quad (38)$$

so that

$$z(t) = \int_c^d G(t, s) f(s) ds \quad (39)$$

is a solution of the inhomogeneous equation

$$z'''(t) + 4tz'(t) + 2z(t) = f(t), \quad c < t < d, \quad (40)$$

such that $z(0) = 0$. That $G(t, s)$ and its partial derivatives $\partial_1 G(t, s)$ and $\partial_2 G(t, s)$ are bounded for all $(t, s) \in \mathbb{R} \times \mathbb{R}$ is clear given the properties of G_0 and Ai .

2.4. Generalized phase functions

Trigonometric and exponential phase functions are particular cases of a more general construction which we now discuss. It can be easily seen that the transformation

$$y(t) = \frac{z(\lambda(t))}{\sqrt{\lambda'(t)}} \quad (41)$$

takes the solutions of

$$z''(t) + \tilde{q}(t)z(t) = 0 \quad (42)$$

to those of (1) provided λ satisfies the nonlinear ordinary differential equation

$$\omega^2 q(t) - \tilde{q}(\lambda(t))(\lambda'(t))^2 + \frac{3}{4} \left(\frac{\lambda''(t)}{\lambda'(t)} \right)^2 - \frac{1}{2} \frac{\lambda'''(t)}{\lambda'(t)} = 0. \quad (43)$$

We will call (43) the generalized Kummer equation and refer to its solutions as generalized phase functions for (1). Their utility lies in the fact that the combination of the solutions of (42) and the transformation λ can be simpler in some fashion than the solutions of (1).

When q is strictly positive, one usually chooses \tilde{q} to be 1. In this event, (43) becomes Kummer's equation (19) and (42) is

$$z''(t) + z(t) = 0. \quad (44)$$

Since $\{\cos(t), \sin(t)\}$ is a basis in the space of solutions of (44), it follows that if α satisfies (19), then (14) is a basis in the space of solutions of (1). That is, the solutions of Kummer's equation are trigonometric phase functions. When q is strictly negative, one typically takes $\tilde{q} = -1$ so that (43) becomes

$$\omega^2 q(t) + (\beta'(t))^2 + \frac{3}{4} \left(\frac{\beta''(t)}{\beta'(t)} \right)^2 - \frac{1}{2} \frac{\beta'''(t)}{\beta'(t)} = 0. \quad (45)$$

Since, in this case, $\{\exp(-t), \exp(t)/2\}$ is a basis in the space of solutions of (42), (16) is a basis in the space of solutions of (1) whenever β satisfies (45). That is, the solutions of (45) are exponential phase functions for (1).

In this paper, our interest is in the case in which the coefficient q in (1) has a simple turning point. In this event, it is more appropriate to take $\tilde{q}(t) = t$ so that (42) becomes Airy's equation (8) and (43) becomes the Airy-Kummer equation (24). Since the Airy functions $\text{Ai}(t)$ and $\text{Bi}(t)$ constitute a basis in the space of solutions of (8), the solutions of (24) are Airy phase functions for (1).

3. Existence of slowly-varying Airy phase functions

In this section, we prove the estimate (22). Our argument is divided into three parts. In Subsection 3.1, we use the method of matched asymptotic expansions to show that, for all sufficiently large ω , there exists a sequence u_0, u_1, \dots, u_M of slowly-varying functions such that when the function γ_M defined via

$$\gamma_M(t) = \omega^{\frac{2}{3}} \left(u_0(t) + \frac{u_1(t)}{\omega^2} + \dots + \frac{u_M(t)}{\omega^{2M}} \right) \quad (46)$$

is inserted into the Airy-Kummer equation, the residual

$$R_M(t) = \omega^2 q(t) - \gamma_M(t)(\gamma'_M(t))^2 + \frac{3}{4} \left(\frac{\gamma''_M(t)}{\gamma'_M(t)} \right)^2 - \frac{1}{2} \frac{\gamma'''_M(t)}{\gamma'_M(t)} \quad (47)$$

and its derivative are on the order of ω^{-2M} . It will emerge during the course of our argument that γ'_M is nonzero on $[a, b]$, so that (47) is well-defined, and we will also show that γ_M has only a single (simple) zero t_0 in $[a, b]$. In Subsection 3.2, we represent a solution γ of the Airy-Kummer equation in the form

$$\gamma(t) = \int_{t_0}^t \gamma'_M(s) \exp(2\delta(s)) ds \quad (48)$$

and derive an integral equation for the function δ . Finally, in Subsection 3.3, we use a contraction mapping argument to show the existence of a solution of this integral equation whose $L^\infty([a, b])$ norm is on the order of $\omega^{-2(M+1)}$. It follows by applying the integral mean value theorem to (48) that, for each $t \in [a, b]$, there exists a point ξ in $[a, b]$ such that

$$\gamma(t) = \left(\int_{t_0}^t \gamma'_M(s) ds \right) \exp(2\delta(\xi)) = \gamma_M(t) \left(1 + 2\delta(\xi) + (\delta(\xi))^2 + \dots \right). \quad (49)$$

The estimate (22) follows immediately from (49) and the fact that the $L^\infty([a, b])$ norm of δ is on the order of $\omega^{-2(M+1)}$.

3.1. Formal asymptotic expansion

It is convenient to define the function

$$Q(t) = \int_0^t \sqrt{|q(s)|} ds. \quad (50)$$

By assumption, q is a smooth, $q(t) \sim t$ as $t \rightarrow 0$ and the only zero of q in the interval $[a, b]$ occurs at the point 0. These properties of q together with l'Hôpital's rule can be used to show that Q and Q' have the forms

$$Q(t) = \frac{2}{3} \text{sign}(t) |t|^{\frac{3}{2}} Q_0(t) \quad \text{and} \quad Q'(t) = \sqrt{|t|} Q_1(t), \quad (51)$$

where Q_0 and Q_1 are smooth and positive on $[a, b]$.

We plug the ansatz (46) into the Airy-Kummer equation (24) and write the resulting expression as

$$\omega^2 q(t) - \omega^2 \left(\sum_{j=0}^M \frac{u_j(t)}{\omega^{2j}} \right) \left(\sum_{j=0}^M \frac{u'_j(t)}{\omega^{2j}} \right)^2 + \sum_{j=0}^{\infty} \frac{S_j(u_0(t), u_1(t), \dots, u_{\min(j, M)}(t))}{\omega^{2j}} = 0, \quad (52)$$

where S_j is the coefficient of ω^{-2j} in the formal expansion of the Schwarzian derivative

$$\frac{3}{4} \left(\frac{\gamma''_M(t)}{\gamma'_M(t)} \right)^2 - \frac{1}{2} \frac{\gamma'''_M(t)}{\gamma'_M(t)} \quad (53)$$

in inverse powers of ω obtained by expanding the parenthetical expression in the formula

$$\frac{1}{\gamma'_M(t)} = \frac{1}{\omega^{\frac{2}{3}} u'_0(t)} \left(1 + \frac{1}{\omega^2} \frac{u'_1(t)}{u'_0(t)} + \frac{1}{\omega^4} \frac{u'_2(t)}{u'_0(t)} + \dots + \frac{1}{\omega^{2M}} \frac{u'_M(t)}{u'_0(t)} \right)^{-1} \quad (54)$$

in a Neumann series, plugging the result into (53) and making use of the obvious expressions for

γ_M'' and γ_M''' obtained from (46). By grouping terms in (52), we obtain an expression of the form

$$\begin{aligned} & \omega^2 F_{-1}(Q(t), u_0(t)) + F_0(Q(t), u_0(t), u_1(t)) + \frac{1}{\omega^2} F_1(Q(t), u_0(t), u_1(t), u_2(t)) + \cdots + \\ & \frac{1}{\omega^{2(M-1)}} F_{M-1}(Q(t), u_0(t), \dots, u_M(t)) + \text{res}_M(t) = 0, \end{aligned} \quad (55)$$

where

$$F_{-1}(Q(t), u_0(t)) = \text{sign}(t) (Q'(t))^2 - (u_0'(t))^2 (u_0(t)), \quad (56)$$

each subsequent F_k takes the form

$$F_k(Q(t), u_0(t), u_1(t), \dots, u_k(t), u_{k+1}(t)) = -(u_0'(t))^2 u_{k+1}(t) - 2u_0(t) u_0'(t) u_{k+1}'(t) + \text{res}_k(t) \quad (57)$$

with

$$\text{res}_k(t) = S_k(u_0(t), \dots, u_k(t)) - \sum_{\substack{i_1+i_2+i_3=k \\ 0 \leq i_1, i_2, i_3 < k}} u_{i_1}(t) u_{i_2}'(t) u_{i_3}'(t) \quad (58)$$

and

$$\text{res}_M(t) = \sum_{j=M}^{\infty} \frac{S_j(u_0(t), \dots, u_M(t))}{\omega^{2j}} - \sum_{k=2M}^{4M} \frac{1}{\omega^{2k}} \sum_{\substack{i_1+i_2+i_3=k \\ 0 \leq i_1, i_2, i_3 \leq M}} u_{i_1}(t) u_{i_2}'(t) u_{i_3}'(t). \quad (59)$$

The nonlinear differential equation $F_{-1}(Q(t), u_0(t)) = 0$ admits the solution

$$u_0(t) = \text{sign}(t) \left(\frac{3}{2} \text{sign}(t) Q(t) \right)^{\frac{2}{3}}. \quad (60)$$

From (51), we see that

$$u_0(t) = t (Q_0(t))^{\frac{2}{3}} \quad \text{and} \quad u_0'(t) = \frac{Q_1(t)}{(Q_0(t))^{\frac{1}{3}}} \quad (61)$$

so that u_0 is smooth, u_0' is nonzero on $[a, b]$ and the only zero of u_0 on $[a, b]$ is the simple zero at 0. With our choice of u_0 , F_k becomes

$$F_k(Q(t), u_0(t), u_1(t), \dots, u_k(t), u_{k+1}(t)) = -2^{\frac{2}{3}} Q'(t) \frac{d}{dt} \left[(3Q(t))^{\frac{1}{3}} u_{k+1}(t) \right] + \text{res}_k(t). \quad (62)$$

We now view $F_0(Q(t), u_0(t), u_1(t)) = 0$ as a first order linear ordinary differential equation defining u_1 . The general solution is

$$u_1(t) = (3Q(t, \omega))^{-\frac{1}{3}} \left(C + 2^{-\frac{2}{3}} \int_0^t \frac{\text{res}_0(s)}{Q'(s)} ds \right) \quad (63)$$

with C an arbitrary constant. We choose $C = 0$ so that u_1 is equal to

$$\frac{2^{-\frac{2}{3}}}{\text{sign}(t) \sqrt{|t|} (3Q_0(t))^{\frac{1}{3}}} \int_0^t \frac{\text{res}_0(s)}{\text{sign}(s) \sqrt{|s|} \sqrt{q_0(s)}} ds. \quad (64)$$

It is easy to see that

$$S_0(u_0(t)) = \frac{3}{4} \left(\frac{u_0''(t)}{u_0'(t)} \right)^2 - \frac{1}{2} \frac{u_0'''(t)}{u_0'(t)} \quad (65)$$

is smooth on $[a, b]$ since $u_0'(t)$ is nonzero there. It follows that $\text{res}_0(t)$ is also smooth. In particular, $u_1(t)$ is of the form

$$\frac{f(t)}{k(t)} \int_0^t \frac{g(s)}{k(s)} ds \quad (66)$$

with f and g smooth functions and $k(s) = \text{sign}(s)\sqrt{|s|}$. It is straightforward, but somewhat tedious, to use l'Hôpital's rule to show that functions of the form (66) are smooth. We note that any other choice of C would result in u_1 being singular at 0. Since u_1 is smooth, u_0 is smooth and u_0' does not vanish on $[a, b]$,

$$S_1(u_0(t), u_1(t)) = -\frac{3 u_1'(t) (u_0''(t))^2}{2 (u_0'(t))^3} + \frac{3 u_1''(t) u_0''(t)}{2 (u_0'(t))^2} + \frac{1 u_0'''(t) u_1'(t)}{2 (u_0'(t))^2} - \frac{1 u_1'''(t)}{2 u_0'(t)} \quad (67)$$

and also res_1 are smooth on $[a, b]$. We can therefore apply the preceding argument to show that there exists a smooth solution u_2 of the first order linear ordinary differential equation

$$F_1(Q(t), u_0(t), u_1(t), u_2(t)) = 0. \quad (68)$$

Continuing in this fashion gives us the desired sequence of smooth functions u_0, u_1, \dots, u_M . By construction, the u_j are independent of ω when q is independent of ω and slowly-varying in the sense discussed in the introduction when q depends on ω but its derivatives are bounded independent of ω .

Now that u_0, u_1, \dots, u_M have been constructed, we will show that the formal calculations used to obtain the expansion of (53) are justified when ω is sufficiently large. To that end, we observe that (46), together with the properties of u_0 discussed above, imply that there is a constant ω_0 such that, for all $\omega \geq \omega_0$, γ_M' is nonzero on $[a, b]$ and γ_M has exactly one zero t_0 in $[a, b]$. It is therefore acceptable to take the reciprocal of γ_M' in (54). Moreover, it is clear that the Neumann series used to expand the parenthetical expression in (54) is convergent for all sufficiently large ω , and we increase ω_0 , if necessary, to ensure that this is the case for all $\omega \geq \omega_0$. It follows that, for all $\omega \geq \omega_0$, the quantity res_M given in (59) is well-defined. By the method of construction of the u_j , res_M is the only nonzero term in (55). Accordingly, the residual R_M obtained when γ_M is inserted into the Airy-Kummer equation is equal in magnitude to res_M . Because res_M is on the order of ω^{-2M} and the derivatives of the u_j are bounded independent of ω , there exists a constant C_1 such that

$$\sup_{a \leq t \leq b} |R_M(t)|_\infty \leq \frac{C_1}{\omega^{2M}} \quad \text{and} \quad \sup_{a \leq t \leq b} |R_M'(t)|_\infty \leq \frac{C_1}{\omega^{2M}} \quad (69)$$

for all $\omega \geq \omega_0$.

Remark 1. *The method used here to construct a formal asymptotic expansion representing a solution of the Airy-Kummer equation (24) can be applied to the generalized Kummer equation (43) when $\tilde{q}(t) = t^\sigma$, $q(t) = t^\sigma q_0(t)$ with q_0 a smooth positive function and σ equal to $-1, 0$ or 1 . Interestingly, though, it fails for all other values of σ . Inserting*

$$\gamma(t) = \omega^{\frac{2}{2+\sigma}} \left(u_0(t) + \frac{u_1(t)}{\omega^2} + \dots + \frac{u_M(t)}{\omega^{2M}} \right) \quad (70)$$

into (43) yields a sequence of differential equations defining the u_j , the first of which is the nonlinear equation

$$\text{sign}(t)(Q'(t))^2 - (u_0'(t))^2 (u_0(t))^\sigma. \quad (71)$$

For all real-valued $\sigma > -2$, (71) admits the smooth solution

$$u_0(t) = \text{sign}(t) \left(\frac{2 + \sigma}{2} \text{sign}(t) \int_0^t \sqrt{|q(s)|} ds \right)^{\frac{2}{2+\sigma}}. \quad (72)$$

However, the subsequent equation which defines u_1 is

$$2^{\frac{2}{2+\sigma}} Q'(t) \frac{d}{dt} \left[((\sigma + 2)Q(t))^{\frac{\sigma}{2+\sigma}} u_1(t) \right] = \text{res}_0(t) := \frac{3}{4} \left(\frac{\gamma_0''(t)}{\gamma_0'(t)} \right)^2 - \frac{1}{2} \left(\frac{\gamma_0'''(t)}{\gamma_0'(t)} \right), \quad (73)$$

and its general solution is

$$u_1(t) = \frac{2^{-\frac{2}{2+\sigma}}}{((\sigma + 2)Q(t))^{\frac{\sigma}{2+\sigma}}} \left(C + \int_0^t \frac{\text{res}_0(s)}{Q'(s)} ds \right), \quad (74)$$

which has the form

$$\frac{f(t)}{k(t)} \left(C + \int_0^t \frac{g(s)}{k(s)} ds \right) \quad (75)$$

with $k(t) \sim |t|^{\frac{\sigma}{2}}$ as $t \rightarrow 0$. It can be readily seen that there is no choice of C which makes (75) smooth for all smooth f and g unless σ is one of the special values $-1, 0$ or 1 .

3.2. Integral equation formulation

We will now reformulate an integral equation for a function δ such that (48) satisfies the Airy-Kummer equation. We will assume throughout this subsection that $\omega \geq \omega_0$, so that γ'_M is nonzero on $[a, b]$ and t_0 is the sole (simple) zero of γ_M in $[a, b]$.

Inserting (48) into the Airy-Kummer equation yields

$$\begin{aligned} \delta''(t) - \frac{\gamma_M''(t)}{\gamma_M'(t)} \delta'(t) + \exp(4\delta(t)) (\gamma_M'(t))^2 \int_{t_0}^t \gamma_M'(s) \exp(2\delta(s)) ds \\ = \omega^2 q(t) + \frac{3}{4} \left(\frac{\gamma_M''(t)}{\gamma_M'(t)} \right)^2 - \frac{1}{2} \frac{\gamma_M'''(t)}{\gamma_M'(t)} + (\delta'(t))^2, \end{aligned} \quad (76)$$

and we subtract $(\gamma_M'(t))^2 \gamma_M(t)$ from both sides of (76) to obtain

$$\begin{aligned} \delta''(t) - \frac{\gamma_M''(t)}{\gamma_M'(t)} \delta'(t) + \exp(4\delta(t)) (\gamma_M'(t))^2 \int_{t_0}^t \gamma_M'(s) \exp(2\delta(s)) ds - (\gamma_M'(t))^2 \gamma_M(t) \\ = R_M(t) + (\delta'(t))^2, \end{aligned} \quad (77)$$

where R_M is given by (47). Expanding the exponentials appearing in (77) in power series and moving all of the terms of order $(\delta(t))^2$ or higher to the right-hand side results in

$$\begin{aligned} \delta''(t) - \frac{\gamma_M''(t)}{\gamma_M'(t)} \delta'(t) + 4\gamma_M(t) (\gamma_M'(t))^2 \delta(t) + 2 (\gamma_M'(t))^2 \int_{t_0}^t \gamma_M'(s) \delta(s) ds \\ = R_M(t) + F(t, \delta(t), \delta'(t)), \end{aligned} \quad (78)$$

where

$$F(t, \delta(t), \delta'(t)) = (\delta'(t))^2 - (\gamma_M'(t))^2 \int_{t_0}^t \gamma_M'(s) \left(\frac{(2\delta(s))^2}{2} + \frac{(2\delta(s))^3}{3!} + \dots \right) ds$$

$$\begin{aligned}
& -4\delta(t) (\gamma'_M(t))^2 \int_{t_0}^t \gamma'_M(s) \left(2\delta(s) + \frac{(2\delta(s))^2}{2} + \dots \right) ds \\
& - \left(\frac{(4\delta(t))^2}{2} + \frac{(4\delta(t))^3}{3!} + \dots \right) (\gamma'_M(t))^2 \int_{t_0}^t \gamma'_M(s) \exp(2\delta(s)) ds.
\end{aligned} \tag{79}$$

We next observe that if z solves the equation

$$(\gamma'_M(t))^2 (z'''(\gamma_M(t)) + 4\gamma_M(t)z'(\gamma_M(t)) + 2(z(\gamma_M(t)) - z(\gamma_M(t_0)))) = f(t), \quad a < t < b, \tag{80}$$

with $f(t) = R_M(t) + F(t, \delta(t), \delta'(t))$, and we let $\delta(t) = z'(\gamma_M(t))$, then δ satisfies (78). Since γ'_M is nonvanishing on $[a, b]$ and $\gamma_M(t_0) = 0$, we can divide by $(\gamma'_M(t))^2$ and introduce the new variable $u = \gamma_M(t)$ in (80) to obtain

$$z'''(u) + 4uz'(u) + 2(z(u) - z(0)) = f(\gamma_M^{-1}(u)) \left(\frac{1}{\gamma'_M(\gamma_M^{-1}(u))} \right)^2, \quad \gamma_M(a) < u < \gamma_M(b). \tag{81}$$

Using the Green's function $G(t, s)$ defined in Section 2.3, we can express a solution of (81) as

$$z(u) = \int_{\gamma_M(a)}^{\gamma_M(b)} G(u, v) f(\gamma_M^{-1}(v)) \left(\frac{1}{\gamma'_M(\gamma_M^{-1}(v))} \right)^2 dv. \tag{82}$$

We note that G was chosen so that the function defined in (82) satisfies the condition $z(0) = 0$. By differentiating (82) and letting $u = \gamma_M(t)$, $v = \gamma_M(s)$ in the resulting expression, we obtain the integral equation

$$\delta(t) = z'(\gamma_M(t)) = \int_a^b K(t, s) (R_M(s) + F(s, \delta(s), \delta'(s))) ds, \tag{83}$$

where

$$K(t, s) = \frac{\partial_1 G(\gamma_M(t), \gamma_M(s))}{\gamma'_M(s)}, \tag{84}$$

R_M is the residual (47) and F is given by (79).

We have arrived at the desired integral equation formulation (83) for the function δ . Our choice of the Green's function $G(t, s)$ ensures that $K(t, s)$ is on the order of ω^{-1} , its derivatives are bounded independent of ω and its antiderivatives are on the order of ω^{-2} . In particular, using (36) and (37), and the formulas derived from them through integration, it can be shown that there exists a constant C_2 such that

$$\sup_{a \leq s, t \leq b} |K(t, s)| \leq \frac{C_2}{\omega} \quad \text{and} \quad \sup_{a \leq s, t \leq b} |\tilde{K}(t, s)| \leq \frac{C_2}{\omega^2}, \tag{85}$$

where

$$\tilde{K}(t, s) = \int_0^t K(t, s) ds. \tag{86}$$

Moreover, the formulas which result from differentiating (36) and (37) imply that we can adjust C_2 such that the bounds

$$\sup_{a \leq s, t \leq b} |\partial_1 K(t, s)| \leq C_2 \quad \text{and} \quad \sup_{a \leq s, t \leq b} |\partial_1 \tilde{K}(t, s)| \leq \frac{C_2}{\omega} \tag{87}$$

hold in addition to (85).

3.3. Contraction mapping argument

In this subsection, we prove that there exists an $r > 0$ such that, for all sufficiently large ω , the nonlinear integral operator

$$T[\delta](t) = \int_a^b K(t, s) (R_M(s) + F(s, \delta(s), \delta'(s))) ds \quad (88)$$

is a contraction on a closed ball of radius $r\omega^{-2M}$ in the Banach space $C^1([a, b])$ endowed with the norm $\|\cdot\|_\omega$ defined in (25). Since

$$\|\delta\|_\infty \leq \frac{r}{\omega^{2M+2}} \quad \text{and} \quad \|\delta'\|_\infty \leq \frac{r}{\omega^{2M+1}} \quad (89)$$

when $\|\delta\|_\omega \leq r\omega^{-2M}$, it will immediately follow that, for all sufficiently large ω , there exists a solution δ of (83) with $\|\delta\|_\infty \leq r\omega^{-2(M+2)}$. We will, once again, assume that $\omega \geq \omega_0$, so that γ_M has the necessary properties and (69) holds. However, we note that this condition is not necessarily sufficient for the argument of this section; i.e., it might be that we need ω to be strictly greater than ω_0 in order for T to be a contraction.

By construction, we have that

$$\gamma_M(t) = \mathcal{O}\left(\omega^{\frac{2}{3}}\right) \quad \text{and} \quad \gamma'_M(t) = \mathcal{O}\left(\omega^{\frac{2}{3}}\right) \quad \text{as } \omega \rightarrow \infty. \quad (90)$$

It follows from (89) and (90) that each of the terms in (79) is on the order of $\omega^{-(4M+2)}$ when $\|\delta\|_\omega \leq r\omega^{-2M}$. In particular, there exists a smooth function E , which is bounded independent of ω , and such that

$$\sup_{a \leq t \leq b} |F(t, \delta(t), \delta'(t))| \leq \frac{E(r)}{\omega^{4M+2}} \quad (91)$$

for all $\|\delta\|_\omega \leq r\omega^{-2M}$. It follows from (85), (87) and (91) that

$$\left| \int_a^b K(t, s) F(s, \delta(s), \delta'(s)) ds \right| \leq \frac{C_2 E(r)}{\omega^{4M+3}} \quad (92)$$

and

$$\left| \int_a^b \partial_1 K(t, s) F(s, \delta(s), \delta'(s)) ds \right| \leq \frac{C_2 E(r)}{\omega^{4M+2}} \quad (93)$$

for all $\|\delta\|_\omega \leq r\omega^{-2M}$. Now integration by parts gives us the formula

$$\int_a^b K(t, s) R_M(s) ds = \tilde{K}(t, b) R_M(b) - \tilde{K}(t, a) R_M(a) - \int_a^b \tilde{K}(t, s) R'_M(s) ds, \quad (94)$$

where \tilde{K} is the antiderivative of K defined in (86), and we combine (69) and (85) with (94) to see that

$$\sup_{a \leq t \leq b} \left| \int_a^b K(t, s) R_M(s) ds \right| \leq \frac{C_1 C_2 (b - a + 2)}{\omega^{2M+2}}. \quad (95)$$

We differentiate (94) and proceed as before, except we use (87) in place of (85), to obtain

$$\sup_{a \leq t \leq b} \left| \int_a^b \partial_1 K(t, s) R_M(s) ds \right| \leq \frac{C_1 C_2 (b - a + 2)}{\omega^{2M+1}}. \quad (96)$$

By combining (92) and (95), we obtain the bound

$$\|T[\delta]\|_\infty \leq \frac{C_1 C_2 (b - a + 2)}{\omega^{2M+2}} + \frac{C_2 E(r)}{\omega^{4M+3}}, \quad (97)$$

which holds for all $\|\delta\|_\omega \leq r\omega^{-2M}$. Likewise, (93) and (96) show that

$$\|T[\delta']\|_\infty \leq \frac{C_1 C_2 (b - a + 2)}{\omega^{2M+1}} + \frac{C_2 E(r)}{\omega^{4M+2}} \quad (98)$$

for all $\|\delta\|_\omega \leq r\omega^{-2M}$. Now we let $r = 2C_1 C_2 (b - a + 2)$ and observe that (97) and (98) imply

$$\|T[\delta]\|_\infty \leq \frac{r}{\omega^{2M+2}} \quad \text{and} \quad \|T[\delta']\|_\infty \leq \frac{r}{\omega^{2M+1}} \quad (99)$$

for all $\|\delta\|_\omega \leq r\omega^{-2M}$ and all sufficiently large ω . That is, T preserves the closed ball of radius $r\omega^{-2M}$ centered at 0 in the space $C^1([a, b])$ endowed with the $\|\cdot\|_\omega$ norm, provided $\omega \geq \omega_0$ is large enough that

$$\frac{E(r)}{\omega^{2M+1}} \leq C_1 (b - a + 2). \quad (100)$$

It remains to show that T is a contraction for sufficiently large ω . To that end, we observe that the Fréchet derivative of T at the point δ is the linear operator

$$D_\delta T[h](t) = \int_a^b K(t, s) H(s, \delta(s), \delta'(s), h(s), h'(s)) ds, \quad (101)$$

where $H(s, \delta(s), \delta'(s), h(s), h'(s))$ is

$$\begin{aligned} & 2\delta'(t)h'(t) - 2(\gamma'_M(t))^2 \int_{t_0}^t \gamma'_M(s) \left(2\delta(s) + \frac{(2\delta(s))^2}{2} + \dots \right) h(s) ds \\ & - 4h(t) (\gamma'_M(t))^2 \int_{t_0}^t \gamma'_M(s) \left(2\delta(s) + \frac{(2\delta(s))^2}{2} + \dots \right) ds \\ & - 8\delta(t) (\gamma'_M(t))^2 \int_{t_0}^t \gamma'_M(s) \exp(2\delta(s)) h(s) ds \\ & - 4 \left(4\delta(t) + \frac{(4\delta(t))^2}{2} + \dots \right) h(t) (\gamma'_M(t))^2 \int_{t_0}^t \gamma'_M(s) \exp(2\delta(s)) ds \\ & - 8 \left(\frac{(4\delta(t))^2}{2} + \frac{(4\delta(t))^3}{3!} + \dots \right) (\gamma'_M(t))^2 \int_{t_0}^t \gamma'_M(s) \exp(2\delta(s)) h(s) ds. \end{aligned} \quad (102)$$

Using (85), (89) and (90), we see that there exists a constant C_3 which is independent of ω and such that

$$\left| \int_{t_0}^t K(t, s) H(s, \delta(s), \delta'(s), h(s), h'(s)) ds \right| \leq \frac{C_3}{\omega^{2M+1}} \|h\|_\infty + \frac{C_3}{\omega^{2M+2}} \|h'\|_\infty \quad (103)$$

and

$$\left| \int_{t_0}^t \partial_1 K(t, s) H(s, \delta(s), \delta'(s), h(s), h'(s)) ds \right| \leq \frac{C_3}{\omega^{2M}} \|h\|_\infty + \frac{C_3}{\omega^{2M+1}} \|h'\|_\infty \quad (104)$$

whenever $\|\delta\|_\omega \leq r\omega^{-2M}$. It follows from (103) and (104) that

$$\|D_\delta T[h]\|_\omega \leq \frac{C_3}{\omega^{2M+1}} \|h\|_\omega \quad (105)$$

for all $\|\delta\|_\omega \leq r\omega^{-2M}$, which shows that T is a contraction on the ball of radius r centered at 0 in $C^1([a, b])$ provided ω is sufficiently large. This completes our proof of the existence of a slowly-varying solution of the Airy-Kummer equation.

4. Convergence of the Newton-Kantorovich Method

In this section, we discretize the Airy-Kummer equation (24) over an interval of the form $[-a_0, a_0]$ via a Chebyshev spectral method and apply Newton-Kantorovich method to the resulting system of nonlinear algebraic equations. We show that if the first order approximate γ_0 is used to form an initial guess, then the Newton iterates converge to a vector representing the slowly-varying solution of the Airy-Kummer equation whose existence was established in the preceding section provided ω is sufficiently large and a_0 is sufficiently small. In the course of our proof, we will see that the number of nodes k used to discretize the Airy-Kummer equation must be relatively small (we take $k = 16$ in the experiments of this paper, and this causes no difficulties). However, because the Chebyshev spectral discretizations of the various functions which arise in the course of the proof converge as a_0 goes to 0, this does not impose a significant limitation on the applicability of our algorithm.

We divide our argument into four parts. In Subsection 4.1, we form the spectral discretization of the Airy-Kummer equation. We then show in Subsection 4.2 that the Fréchet derivative of the operator in question is invertible provided a_0 is sufficiently small and ω sufficiently large. In Subsection 4.3, we give a Lipschitz bound on the Fréchet derivative. Finally, we conclude our proof by applying the Newton-Kantorovich theorem in Subsection 4.4.

4.1. Spectral discretization of the Airy-Kummer equation

We form a spectral discretization of the Airy-Kummer equation (24) by representing the unknown solution γ and the coefficient q via the vectors

$$\boldsymbol{\gamma} = \left(\gamma(t_0) \quad \gamma(t_1) \quad \cdots \quad \gamma(t_k) \right)^\top \quad (106)$$

and

$$\boldsymbol{q} = \left(q(t_1) \quad q(t_2) \quad \cdots \quad q(t_k) \right)^\top \quad (107)$$

of their values at the extremal Chebyshev nodes t_1, \dots, t_k on the interval $[-a_0, a_0]$, replacing the differential operators with Chebyshev spectral differentiation matrices and requiring that the resulting system of semidiscrete equations hold at the nodes t_1, \dots, t_k . This procedure yields the system of nonlinear algebraic equations $R(\boldsymbol{\gamma}) = 0$, where R is the mapping $\mathbb{R}^k \rightarrow \mathbb{R}^k$ given by the matrix

$$R(\boldsymbol{\gamma}) = \omega^2 \boldsymbol{q} - \boldsymbol{\gamma} \circ \left(\frac{\mathcal{D}_k}{a_0} \boldsymbol{\gamma} \right)^{\circ 2} + \frac{3}{4} \left(\left(\frac{\mathcal{D}_k}{a_0} \right)^2 \boldsymbol{\gamma} \right)^{\circ 2} \circ \left(\frac{\mathcal{D}_k}{a_0} \boldsymbol{\gamma} \right)^{\circ -2} - \frac{1}{2} \left(\left(\frac{\mathcal{D}_k}{a_0} \right)^3 \boldsymbol{\gamma} \right) \circ \left(\frac{\mathcal{D}_k}{a_0} \boldsymbol{\gamma} \right)^{\circ -1}. \quad (108)$$

The Fréchet derivative of R at \mathbf{x} is the linear mapping $\mathbb{R}^k \rightarrow \mathbb{R}^k$ given by the matrix

$$\begin{aligned}
D_{\mathbf{x}}R = & - \left(\text{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{x} \right) \right)^2 - 2 \text{diag}(\mathbf{x}) \text{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{x} \right) \frac{\mathcal{D}_k}{a_0} \\
& - \frac{3}{2} \left(\text{diag} \left(\left(\frac{\mathcal{D}_k}{a_0} \right)^2 \mathbf{x} \right) \right)^2 \left(\text{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{x} \right) \right)^{-2} \frac{\mathcal{D}_k}{a_0} \\
& + \frac{1}{2} \text{diag} \left(\left(\frac{\mathcal{D}_k}{a_0} \right)^3 \mathbf{x} \right) \left(\text{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{x} \right) \right)^{-2} \frac{\mathcal{D}_k}{a_0} \\
& + \frac{3}{2} \text{diag} \left(\left(\frac{\mathcal{D}_k}{a_0} \right)^2 \mathbf{x} \right) \left(\text{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{x} \right) \right)^{-2} \left(\frac{\mathcal{D}_k}{a_0} \right)^2 \\
& - \frac{1}{2} \left(\text{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{x} \right) \right)^{-1} \left(\frac{\mathcal{D}_k}{a_0} \right)^3.
\end{aligned} \tag{109}$$

We let γ_0 be the spectral discretization

$$\gamma_0 = (\gamma_0(t_1) \quad \gamma_0(t_2) \quad \cdots \quad \gamma_0(t_k))^\top \tag{110}$$

of the first order asymptotic approximation γ_0 defined via (7). Owing to our assumptions on q , there exists a smooth function τ such that $\tau(t) = \mathcal{O}(t^2)$ as $t \rightarrow 0$ and $\gamma_0(t) = \omega^{\frac{2}{3}}(t + \tau(t))$. We now write

$$\gamma_0 = \omega^{\frac{2}{3}}(a_0 \mathbf{t} + \boldsymbol{\tau} + \boldsymbol{\epsilon}_1) \quad \text{and} \quad \frac{\mathcal{D}_k}{a_0} \gamma_0 = \omega^{\frac{2}{3}}(\mathbf{1} + \boldsymbol{\tau}' + \boldsymbol{\epsilon}_2), \tag{111}$$

where:

- $\mathbf{t} = (t_1^{\text{cheb}} \quad t_2^{\text{cheb}} \quad \cdots \quad t_k^{\text{cheb}})^\top$ and $\mathbf{1} = (1 \quad 1 \quad \cdots \quad 1)^\top$;

- $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ are the spectral discretizations

$$\boldsymbol{\tau} = (\tau(t_1) \quad \tau(t_2) \quad \cdots \quad \tau(t_k))^\top \quad \text{and} \quad \boldsymbol{\tau}' = (\tau'(t_1) \quad \tau'(t_2) \quad \cdots \quad \tau'(t_k))^\top \tag{112}$$

of the function τ and τ' ; and

- $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ are vectors which account for discretization error.

Since $\tau(t) = \mathcal{O}(t^2)$, we have

$$\|\boldsymbol{\tau}\|_\infty = \mathcal{O}(a_0^2) \quad \text{and} \quad \|\boldsymbol{\tau}'\|_\infty = \mathcal{O}(a_0) \quad \text{as} \quad a_0 \rightarrow 0, \tag{113}$$

while the infinite differentiability of γ_0 implies

$$\|\boldsymbol{\epsilon}_1\|_\infty = \mathcal{O}(a_0^k) \quad \text{and} \quad \|\boldsymbol{\epsilon}_2\|_\infty = \mathcal{O}(a_0^{k-1}) \quad \text{as} \quad a_0 \rightarrow 0. \tag{114}$$

Moreover, because of the form of γ_0 and our assumption that the derivatives of q are bounded independent of ω , the asymptotic estimates (113) and (114) hold uniformly in ω . We close this subsection by noting that, because the residual R_0 defined in (47) is bounded independent of ω and $R(\gamma_0)$ discretizes this quantity, we will have

$$R(\gamma_0) = \mathcal{O}(1) \quad \text{as} \quad \omega \rightarrow \infty, \tag{115}$$

for any fixed a_0 which is sufficiently small so that the spectral discretization of $R(\gamma_0)$ is accurate.

4.2. The Fréchet derivative of R at γ_0

It follows from (109) and (111) that the matrix representing the Fréchet derivative of R at γ_0 can be written in the form

$$D_{\gamma_0}R = -\omega^{\frac{4}{3}} (\mathcal{I}_k + 2 \operatorname{diag}(\mathbf{t})\mathcal{D}_k + S) + T, \quad (116)$$

where

$$\begin{aligned} S = & (\operatorname{diag}(\boldsymbol{\tau}'))^2 + (\operatorname{diag}(\boldsymbol{\epsilon}_2))^2 + 2 \operatorname{diag}(\boldsymbol{\tau}') \operatorname{diag}(\boldsymbol{\epsilon}_2) + 2 \operatorname{diag}(\boldsymbol{\tau}') + 2 \operatorname{diag}(\boldsymbol{\epsilon}_2) + \\ & 2 \operatorname{diag}(\mathbf{t}) \operatorname{diag}(\boldsymbol{\tau}')\mathcal{D}_k + 2 \operatorname{diag}(\mathbf{t}) \operatorname{diag}(\boldsymbol{\epsilon}_2)\mathcal{D}_k + \\ & 2 \operatorname{diag}(\boldsymbol{\tau}) \frac{\mathcal{D}_k}{a_0} + 2 \operatorname{diag}(\boldsymbol{\tau}) \operatorname{diag}(\boldsymbol{\tau}') \frac{\mathcal{D}_k}{a_0} + 2 \operatorname{diag}(\boldsymbol{\tau}) \operatorname{diag}(\boldsymbol{\epsilon}_2) \frac{\mathcal{D}_k}{a_0} + \\ & 2 \operatorname{diag}(\boldsymbol{\epsilon}_1) \frac{\mathcal{D}_k}{a_0} + 2 \operatorname{diag}(\boldsymbol{\epsilon}_1) \operatorname{diag}(\boldsymbol{\tau}') \frac{\mathcal{D}_k}{a_0} + 2 \operatorname{diag}(\boldsymbol{\epsilon}_1) \operatorname{diag}(\boldsymbol{\epsilon}_2) \frac{\mathcal{D}_k}{a_0} \end{aligned} \quad (117)$$

and the operator norm of the matrix T is bounded independent of ω . The matrix $\mathcal{I}_k + 2 \operatorname{diag}(\mathbf{t})\mathcal{D}_k$ is invertible and well-conditioned as long as k is of moderate size. From (113) and (114), it is clear that the operator norm of S goes to 0 as $a_0 \rightarrow 0$. It follows from this and a standard Neumann series argument that the operator

$$\mathcal{I}_k + 2 \operatorname{diag}(\mathbf{t})\mathcal{D}_k + S \quad (118)$$

is invertible for all sufficiently small a_0 . Since the operator norm of S is bounded independent of ω , a second routine Neumann series argument implies that $D_{\gamma_0}R$ is invertible provided (118) is invertible and ω is sufficiently large. Moreover, it is clear from (116) that

$$\left\| (D_{\gamma_0}R)^{-1} \right\|_{\infty} = \mathcal{O} \left(\omega^{-\frac{4}{3}} \right) \text{ as } \omega \rightarrow \infty. \quad (119)$$

4.3. A Lipschitz bound for $\|D_{\mathbf{x}}R - D_{\mathbf{y}}R\|_{\infty}$

From (109), we see that the Fréchet derivative of R at \mathbf{v} can be written as

$$D_{\mathbf{v}}R = \sum_{j=0}^3 \operatorname{diag} (E_j(\mathbf{v})) \left(\frac{\mathcal{D}_k}{a_0} \right)^j, \quad (120)$$

where the E_j are the nonlinear mappings $\mathbb{R}^k \rightarrow \mathbb{R}^k$ defined via the following formulas:

$$\begin{aligned} E_0(\mathbf{v}) &= - \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right)^{\circ 2}, \\ E_1(\mathbf{v}) &= -2\mathbf{v} \circ \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right) - \frac{3}{2} \left(\left(\frac{\mathcal{D}_k}{a_0} \right)^2 \mathbf{v} \right)^{\circ 2} \circ \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right)^{\circ -3} + \frac{1}{2} \left(\left(\frac{\mathcal{D}_k}{a_0} \right)^3 \mathbf{v} \right) \circ \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right)^{\circ -2}, \\ E_2(\mathbf{v}) &= \frac{3}{2} \left(\left(\frac{\mathcal{D}_k}{a_0} \right)^2 \mathbf{v} \right) \circ \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right)^{\circ -2} \quad \text{and} \\ E_3(\mathbf{v}) &= -\frac{1}{2} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right)^{\circ -1}. \end{aligned} \quad (121)$$

Given any $\eta > 0$, it follows from (120) that

$$\begin{aligned} \|D_{\mathbf{x}}R - D_{\mathbf{y}}R\|_{\infty} &\leq \sum_{j=0}^3 \left(\|E_j(\mathbf{x}) - E_j(\mathbf{y})\|_{\infty} \left\| \frac{\mathcal{D}_k}{a_0} \right\|_{\infty}^j \right) \\ &\leq \left(\sum_{j=0}^3 \left\| \frac{\mathcal{D}_k}{a_0} \right\|_{\infty}^j \sup_{\mathbf{v} \in B_{\eta}(\boldsymbol{\gamma}_0)} \|D_{\mathbf{v}}E_j\|_{\infty} \right) \|\mathbf{x} - \mathbf{y}\|_{\infty} \end{aligned} \quad (122)$$

whenever $\mathbf{x}, \mathbf{y} \in B_{\eta}(\boldsymbol{\gamma}_0)$. Simple calculations shows that the Fréchet derivatives of the E_j are given by the matrices

$$\begin{aligned} D_{\mathbf{v}}E_0 &= -2 \operatorname{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right) \frac{\mathcal{D}_k}{a_0}, \\ D_{\mathbf{v}}E_1 &= -2 \left(\operatorname{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right) + \operatorname{diag}(\mathbf{v}) \frac{\mathcal{D}_k}{a_0} \right) \\ &\quad - \frac{3}{2} \left(-3 \left(\operatorname{diag} \left(\left(\frac{\mathcal{D}_k}{a_0} \right)^2 \mathbf{v} \right) \right)^2 \left(\operatorname{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right) \right)^{-4} \frac{\mathcal{D}_k}{a_0} \right. \\ &\quad \left. + 2 \operatorname{diag} \left(\left(\frac{\mathcal{D}_k}{a_0} \right)^2 \mathbf{v} \right) \left(\operatorname{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right) \right)^{-3} \left(\frac{\mathcal{D}_k}{a_0} \right)^2 \right) \\ &\quad + \frac{1}{2} \left(-2 \operatorname{diag} \left(\left(\frac{\mathcal{D}_k}{a_0} \right)^3 \mathbf{v} \right) \left(\operatorname{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right) \right)^{-3} \frac{\mathcal{D}_k}{a_0} + \left(\operatorname{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right) \right)^{-2} \left(\frac{\mathcal{D}_k}{a_0} \right)^3 \right), \\ D_{\mathbf{v}}E_2 &= \frac{3}{2} \left(-2 \operatorname{diag} \left(\left(\frac{\mathcal{D}_k}{a_0} \right)^2 \mathbf{v} \right) \left(\operatorname{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right) \right)^{-3} \frac{\mathcal{D}_k}{a_0} + \left(\operatorname{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right) \right)^{-2} \left(\frac{\mathcal{D}_k}{a_0} \right)^2 \right) \quad \text{and} \\ D_{\mathbf{v}}E_3 &= \frac{1}{2} \left(\operatorname{diag} \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right) \right)^{-2} \frac{\mathcal{D}_k}{a_0}. \end{aligned} \quad (123)$$

It follows from (111), (113) and (114) that for sufficiently small a_0 , we can choose a constant C_3 such that $\|\boldsymbol{\gamma}_0\|_{\infty} \leq C_3 \omega^{\frac{2}{3}}$ and each entry v_j of the vector $\frac{\mathcal{D}_k}{a_0} \boldsymbol{\gamma}_0$ satisfies the inequality

$$\frac{C_3}{2} \omega^{\frac{2}{3}} \leq v_j \leq C_3 \omega^{\frac{2}{3}}. \quad (124)$$

It follows that if

$$\eta < \frac{C_3 a_0}{4 \|\mathcal{D}_k\|_{\infty}} \omega^{\frac{2}{3}}, \quad (125)$$

then

$$\|\mathbf{v}\|_{\infty} \leq \left(1 + \frac{a_0}{4 \|\mathcal{D}_k\|_{\infty}} \right) C_3 \omega^{\frac{2}{3}}, \quad \left\| \frac{\mathcal{D}_k}{a_0} \mathbf{v} \right\|_{\infty} \leq \frac{5C_3}{4} \omega^{\frac{2}{3}} \quad \text{and} \quad \left\| \left(\frac{\mathcal{D}_k}{a_0} \mathbf{v} \right)^{\circ-1} \right\|_{\infty} \leq \frac{C_3}{4} \omega^{\frac{2}{3}} \quad (126)$$

for all $\mathbf{v} \in B_{\eta}(\boldsymbol{\gamma}_0)$. Now (123) and (126) imply that for all $\mathbf{v} \in B_{\eta}(\boldsymbol{\gamma}_0)$,

$$\|D_{\mathbf{v}}E_0\|_{\infty} = \mathcal{O} \left(\omega^{\frac{2}{3}} \right) \quad \text{and} \quad \|D_{\mathbf{v}}E_1\|_{\infty} = \mathcal{O} \left(\omega^{\frac{2}{3}} \right) \quad \text{as } \omega \rightarrow \infty \quad (127)$$

while

$$\|D_{\mathbf{v}}E_2\|_{\infty} = \mathcal{O} \left(\omega^{-\frac{4}{3}} \right) \quad \text{and} \quad \|D_{\mathbf{v}}E_3\|_{\infty} = \mathcal{O} \left(\omega^{-\frac{4}{3}} \right) \quad \text{as } \omega \rightarrow \infty. \quad (128)$$

From (122), (127) and (128), we see that

$$\|D_{\mathbf{x}}R - D_{\mathbf{y}}R\|_{\infty} = \mathcal{O}\left(\omega^{\frac{2}{3}}\right) \|\mathbf{x} - \mathbf{y}\|_{\infty} \quad \text{as } \omega \rightarrow \infty \quad (129)$$

for all \mathbf{x} and \mathbf{y} in $B_{\eta}(\boldsymbol{\gamma}_0)$ provided (125) holds.

4.4. Application of the Newton-Kantorovich theorem

We are now in a position to apply the Newton-Kantorovich theorem. We choose a_0 to be sufficiently small that (115) holds, the operator (118) is invertible and such that (124) holds. From the discussion in the preceding subsection, we know that if η satisfies (125), then the entries of the vector $\frac{\mathcal{D}_k}{a_0}\mathbf{v}$ are bounded away from 0 for all $\mathbf{v} \in B_{\eta}(\boldsymbol{\gamma}_0)$. It follows from this and (108) that the mapping R is continuously differentiable on $B_{\eta}(\boldsymbol{\gamma}_0)$, and we take the open set Ω in Theorem 1 to be $B_{\eta}(\boldsymbol{\gamma}_0)$. Condition (2) of the theorem is obviously satisfied. We showed in Subsection 4.2 that the Fréchet derivative of R at $\boldsymbol{\gamma}_0$ is invertible provided ω is sufficiently large; that is to say, condition (1) of the theorem is satisfied provided ω is large enough.

We now choose the parameters η and λ as follows:

$$\lambda = \left\| (D_{\boldsymbol{\gamma}_0}R)^{-1} R(\boldsymbol{\gamma}_0) \right\|_{\infty} \quad \text{and} \quad \eta = \omega^{\frac{1}{3}}. \quad (130)$$

The inequality (125) clearly holds for sufficiently large ω , and our choice of λ means that condition (4) of the theorem is obviously satisfied. Now, combining (115) and (119) yields

$$\lambda = \mathcal{O}\left(\omega^{-\frac{4}{3}}\right) \quad \text{as } \omega \rightarrow \infty, \quad (131)$$

and it is immediate from (125) and (131) that condition (3) of the theorem is satisfied for sufficiently large ω . It remains only to show that condition (5) holds. Combining (119) and (129) shows that

$$\left\| (D_{\boldsymbol{\gamma}_0}R)^{-1} (D_{\mathbf{x}}R - D_{\mathbf{y}}R) \right\|_{\infty} = \mathcal{O}\left(\omega^{-\frac{2}{3}}\right) \|\mathbf{x} - \mathbf{y}\|_{\infty} \quad \text{as } \omega \rightarrow \infty \quad (132)$$

whenever \mathbf{x} and \mathbf{y} are elements of $B_{\eta}(\boldsymbol{\gamma}_0)$. It follows from this and our choice of η that condition (5) is satisfied for sufficiently large ω .

Having established that all of the requirements of Theorem 1 are satisfied, it now follows that, when ω is sufficiently large, there is a unique solution \mathbf{x}^* of $R(\mathbf{x}) = 0$ in the ball $B_{\eta}(\boldsymbol{\gamma}_0)$, and the sequence $\{\mathbf{x}_j\}$ of Newton-Kantorovich iterates generated by the initial guess $\mathbf{x}_0 = \boldsymbol{\gamma}_0$ converge to \mathbf{x}^* . Because the slowly-varying solution of the Airy-Kummer equation whose existence was established in Section 3 converges to $\boldsymbol{\gamma}_0$ as $\omega \rightarrow \infty$, the vector $\boldsymbol{\gamma}^*$ discretizing it will lie in the ball $B_{\eta}(\boldsymbol{\gamma}_0)$ for all sufficiently large ω . Moreover, since the equation $R(\mathbf{x}) = 0$ discretizes the Airy-Kummer equation which the slowly-varying solution satisfies, $R(\boldsymbol{\gamma}^*) \approx 0$. It follows that \mathbf{x}^* , which is the unique solution of the discretized equation in the ball $B_{\eta}(\boldsymbol{\gamma}_0)$, must agree closely with the discretization $\boldsymbol{\gamma}^*$ of the slowly-varying solution. Finally, we observe that (32) together with (130) and (131) give us the estimate

$$\|\mathbf{x}_j - \boldsymbol{\gamma}^*\|_{\infty} = \mathcal{O}\left(\omega^{\frac{1}{3} - \frac{5}{3}2^j}\right) \quad \text{as } \omega \rightarrow \infty \quad (133)$$

on the rate at which the Newton iterates converge to $\mathbf{x}^* \approx \boldsymbol{\gamma}^*$.

Remark 2. Given any $\epsilon > 0$, we can choose $\eta = \omega^{\frac{2}{3} - \epsilon}$ and the proof given here is still valid. In this event,

$$\|\mathbf{x}_j - \boldsymbol{\gamma}^*\|_{\infty} = \mathcal{O}\left(\omega^{(\frac{2}{3} - \epsilon) - 2^j(2 - \epsilon)}\right) \quad \text{as } \omega \rightarrow \infty. \quad (134)$$

It seems likely that if constants are carefully tracked and accounted for, then it will emerge that the parameter η can be chosen to be on the order of $\omega^{\frac{2}{3}}$. If this is the case, then the Newton iterates converge to γ^* at the rate $\omega^{\frac{2}{3}-2^{j+1}}$.

5. Numerical construction of Airy phase functions

In this section, we describe our numerical method for calculating the slowly-varying Airy phase function γ whose existence was established in Section 3. Our algorithm proceeds in three stages. In the first stage, we numerically compute the values of the approximation γ_0 defined via (7) at the nodes of the k -point Chebyshev extremal grid on the small interval $[-a_0, a_0]$. In the second stage, we use γ_0 to form an initial guess for Newton iterations which converge to a vector giving the values of the desired slowly-varying solution γ at the Chebyshev extremal nodes in $[-a_0, a_0]$. The first-order approximate γ_0 is used to form an initial guess because, unlike its higher order analogues γ_M , it can be readily calculated. In the final stage, we apply a standard adaptive Chebyshev spectral method, with the computed values γ and its derivatives at 0 used as initial conditions, to compute a piecewise Chebyshev expansion representing γ over the entire solution domain $[a, b]$. We detail each of these three procedures in a subsection below.

It will be convenient to write the coefficient $q(t)$ of (1) in the form $tq_0(t)$, where q_0 is smooth and positive on $[a, b]$. Our algorithm takes the following as inputs:

- an interval $[a, b]$ containing 0 and over which the Airy-Kummer phase function γ is to be computed;
- a sufficiently small subinterval of the form $[-a_0, a_0]$ of $[a, b]$ containing 0 on which to apply the Newton-Kantorovich method;
- a positive integer k which controls the order of the Chebyshev expansions used to represent γ ;
- a positive real number ϵ specifying the desired precision for the calculations;
- the value of the parameter ω ; and
- an external subroutine for evaluating q_0 .

It outputs $(k-1)^{st}$ order piecewise Chebyshev expansions representing the Airy phase function γ and its first two derivatives on the interval $[a, b]$. To be entirely clear, a $(k-1)^{st}$ order piecewise Chebyshev expansion on the interval $[a, b]$ is an expansion of the form

$$\begin{aligned} & \sum_{i=1}^{m-1} \chi_{[\xi_{i-1}, \xi_i)}(t) \sum_{j=0}^{k-1} c_{ij} T_j \left(\frac{2}{\xi_i - \xi_{i-1}} t - \frac{\xi_i + \xi_{i-1}}{\xi_i - \xi_{i-1}} \right) \\ & + \chi_{[\xi_{m-1}, \xi_m]}(t) \sum_{j=0}^{k-1} c_{mj} T_j \left(\frac{2}{\xi_m - \xi_{m-1}} t - \frac{\xi_m + \xi_{m-1}}{\xi_m - \xi_{m-1}} \right) \end{aligned} \tag{135}$$

where $a = \xi_0 < \xi_1 < \dots < \xi_m = b$ is a partition of $[a, b]$, χ_I is the characteristic function on the interval I and T_j is the Chebyshev polynomial of degree j . We note that the characteristic function of a half-open interval appears in the first line of (135), whereas the characteristic function of a closed interval appears in the second line. This ensures that exactly one of the characteristic functions appearing in (135) is nonzero for each point t in $[a, b]$.

Once the piecewise Chebyshev expansions representing γ and its first two derivatives have been constructed, both elements of a basis in the space of solutions of (1) and their first derivatives can be readily evaluated at any point in $[a, b]$ at a cost which is independent of ω . Of course, it follows that a large class of initial and boundary value problems for (1) can be solved in time independent of ω and the obtained solutions can be evaluated at arbitrary points in $[a, b]$ in time independent of ω .

5.1. Numerical evaluation of the asymptotic approximation

To evaluate γ_0 , we first compute the values of the function

$$\int_0^t \sqrt{|q(s)|} ds = \int_0^t \sqrt{|s|} \sqrt{q_0(s)} ds \quad (136)$$

at the nodes t_1, \dots, t_k of the Chebyshev extremal grid on $[-a_0, a_0]$. Because of the singularity in the integrand, (136) cannot be evaluated efficiently using Clenshaw-Curtis or Gauss-Legendre quadrature rules. We could use a Gauss-Jacobi rule, but we prefer another technique based on monomial expansions. Although the numerical use of monomial expansions has historically been viewed with suspicion owing to concerns regarding numerical stability, it is shown in [14] that under widely-applicable conditions they are as stable as an orthogonal polynomial basis when used for interpolation.

We proceed by numerically calculating the coefficients of a monomial expansion

$$p(t) = \sum_{j=0}^{k-1} c_j \left(\frac{t}{a_0} \right)^{j-1} \quad (137)$$

representing the smooth function $\sqrt{q_0(t)}$ over $[-a_0, a_0]$. This is done by solving the Vandermonde system which results from enforcing the conditions

$$p(t_i) = \sqrt{q_0(t_i)}, \quad i = 1, \dots, k. \quad (138)$$

Inserting the expansion (137) into (136) and evaluating the integral yields

$$\sum_{j=0}^{k-1} c_j \int_0^t \sqrt{|s|} \left(\frac{s}{a_0} \right)^{j-1} ds = a_0 \sqrt{|t|} \sum_{j=0}^{k-1} \frac{c_j}{j + \frac{1}{2}} \left(\frac{t}{a_0} \right)^j. \quad (139)$$

Using (139), the values of the function (136) at the Chebyshev extremal nodes t_1, \dots, t_k can be easily computed and, once this has been done, it is trivial to evaluate the function γ_0 defined via (7) at those same nodes. The output of the procedure of this subsection is the vector

$$\boldsymbol{\gamma}_0 \approx \left(\gamma_0(t_1) \quad \gamma_0(t_2) \quad \cdots \quad \gamma_0(t_k) \right)^\top \quad (140)$$

approximating the values of the first order approximate γ_0 at the extremal Chebyshev nodes on $[-a_0, a_0]$.

Remark 3. *It is easy to generalize the procedure of this subsection to a nonsymmetric interval of the form $[a_0, b_0]$ containing 0. In this case, for the sake of numerical stability, it is best to use a monomial expansion of the form*

$$p(t) = \sum_{j=0} c_j \left(\frac{2}{b_0 - a_0} t - \frac{b_0 + a_0}{b_0 - a_0} \right)^{j-1} \quad (141)$$

to represent $\sqrt{q_0(t)}$ and the integrals which need to be evaluated are of the form

$$\int_0^t \sqrt{|s|} \left(s - \frac{a_0 + b_0}{2} \right)^{j-1} ds. \quad (142)$$

An explicit expression for (142) can be written in terms of the Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$, which can be easily evaluated via a three-term linear recurrence relation. The necessary formulas can be found, for example, in Chapter 2 of [2].

5.2. Computation of γ over the interval $[-a_0, a_0]$

In this subsection, we describe our procedure for calculating the values of the desired slowly-varying phase function γ over the small interval $[-a_0, a_0]$. Our method consists of using the Newton-Kantorovich method to solve the discretized version $R(\gamma) = 0$ of the Airy-Kummer equation.

We use γ to denote the current iterate and, in the first instance, we take it to be equal to the vector γ_0 . We let \mathbf{q} be the vector defined in (107) and then form the vectors

$$\gamma' = \frac{1}{a_0} \mathcal{D}_k \gamma, \quad \gamma'' = \frac{1}{a_0} \mathcal{D}_k \gamma' \quad \text{and} \quad \gamma''' = \frac{1}{a_0} \mathcal{D}_k \gamma''$$

that give the values of the first, second and third derivatives of the Chebyshev expansion represented via γ at the Chebyshev nodes. Next, we repeatedly perform the following steps:

1. Form the vector $\mathbf{r} = R(\gamma)$, where R is defined via (108).
2. Form the matrix representing the Fréchet derivative $D_\gamma R$ of the operator R defined in (108) at the point γ . It is defined via Formula (109).
3. Solve the system of linear equations $D_\gamma R(\mathbf{h}) = -\mathbf{r}$.
4. Compute the quantity $\zeta = \|\mathbf{h}\|_\infty / \|\gamma\|_\infty$.
5. Let $\gamma = \gamma + \mathbf{h}$, $\gamma' = \frac{1}{a_0} \mathcal{D}_k \gamma$, $\gamma'' = \frac{1}{a_0} \mathcal{D}_k \gamma'$ and $\gamma''' = \frac{1}{a_0} \mathcal{D}_k \gamma''$.
6. If $\zeta > \epsilon$, goto step 1. Otherwise, the procedure terminates.

Upon termination of the above procedure, we have the values of Chebyshev expansions representing the desired slowly-varying Airy phase function γ and its first three derivatives on the interval $[-a_0, a_0]$. We now use these expansions to calculate approximations of $\gamma(0)$, $\gamma'(0)$ and $\gamma''(0)$, and these quantities comprise the output of this stage of our algorithm.

5.3. Extension of γ to $[a, b]$

We now use a standard adaptive Chebyshev spectral method to solve the Airy-Kummer equation over the entire interval $[a, b]$. We impose the conditions that the obtained solution and its first two derivatives agree with the values of $\gamma(0)$, $\gamma'(0)$ and $\gamma''(0)$ computed in the preceding stage of our algorithm.

We describe the solver's operation in the case of the more general problem

$$\begin{cases} \mathbf{y}'(t) &= F(t, \mathbf{y}(t)), & a < t < b, \\ \mathbf{y}(0) &= \mathbf{v} \end{cases} \quad (143)$$

where $F : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is smooth and $\mathbf{v} \in \mathbb{R}^n$. Obviously, the initial value problem for the Airy-Kummer equation we seek to solve can be put into the form (143). The spectral solver

outputs n piecewise $(k - 1)^{st}$ order Chebyshev expansions, one for each of the components $y_i(t)$ of the solution \mathbf{y} of (143).

The solver proceeds in two stages. In the first, it constructs the solution over the interval $[0, b]$. During this stage, two lists of subintervals of $[0, b]$ are maintained: one consisting of what we term "accepted subintervals" and the other of subintervals which have yet to be processed. A subinterval is accepted if the solution is deemed to be adequately represented by a $(k - 1)^{st}$ order Chebyshev expansion on that subinterval. Initially, the list of accepted subintervals is empty and the list of subintervals to process contains the single interval $[0, b]$. The solver proceeds as follows until the list of subintervals to process is empty:

1. Find, in the list of subintervals to process, the interval $[c, d]$ such that c is as small as possible and remove this subinterval from the list
2. Solve the initial value problem

$$\begin{cases} \mathbf{u}'(t) = F(t, \mathbf{u}(t)), & c < t < d \\ \mathbf{u}(c) = \mathbf{w} \end{cases} \quad (144)$$

If $[c, d] = [0, b]$, then we take $\mathbf{w} = \mathbf{v}$. Otherwise, the value of the solutions at the point c has already been approximated, and we use that estimate for \mathbf{w} in (144). If the problem is linear, a straightforward Chebyshev integral equation method is used to solve (144). Otherwise, the trapezoidal method is first used to produce an initial approximation \mathbf{y}_0 of the solution and then Newton's method is applied to refine it. The linearized problems are solved using a Chebyshev integral equation method. In any event, the result is a set of $(k - 1)^{st}$ order Chebyshev expansions

$$u_i(t) \approx \sum_{j=0}^{k-1} c_{ij} T_j \left(\frac{2}{d-c} t - \frac{d+c}{d-c} \right), \quad i = 1, \dots, n \quad (145)$$

which purportedly approximate the components u_1, \dots, u_n of the solution of (144).

3. Compute the quantities

$$\frac{\sqrt{\sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^{k-1} |c_{ij}|^2}}{\sqrt{\sum_{j=0}^{k-1} |c_{ij}|^2}}, \quad i = 1, \dots, n \quad (146)$$

where c_{ij} are the coefficients in the expansions (145). If any of the resulting values is larger than ϵ , then split the subinterval into two halves $(c, \frac{c+d}{2})$ and $(\frac{c+d}{2}, d)$ and place them on the list of subintervals to process. Otherwise, place the subinterval (c, d) on the list of accepted subintervals.

At the conclusion of this stage, we have $(k - 1)^{st}$ order piecewise Chebyshev expansions representing each component of the solution over the interval $[0, b]$, with the list of accepted subintervals determining the partition of $[0, b]$ associated with the piecewise expansions.

In its second stage, an analogous procedure is used to construct piecewise Chebyshev expansions representing the solution over the interval $[a, 0]$. In each step, instead of choosing the unprocessed interval $[c, d]$ such that c is as small as possible and solving an initial value problem over $[c, d]$, a

terminal value problem of the form

$$\begin{cases} \mathbf{u}'(t) = F(t, \mathbf{u}(t)), & c < t < d \\ \mathbf{u}(d) = \mathbf{w}, \end{cases} \quad (147)$$

where $[c, d]$ is the unprocessed interval such that d is as large as possible, is solved. At the conclusion of this second stage, we have a $(k - 1)^{st}$ order piecewise Chebyshev expansion representing each component of the solution over the interval $[a, 0]$, with the list of accepted subintervals determining the partition of $[a, 0]$ associated with the piecewise expansions. Obviously, amalgamating the piecewise expansions of the u_j produced during the two stages gives us the desired piecewise Chebyshev expansions of the components of the solution of (143) over the entire interval $[a, b]$.

6. Numerical Experiments

In this section, we present the results of numerical experiments conducted to illustrate the properties of our method. We implemented our algorithm in Fortran and compiled our code with version 14.2.1 of the GNU Fortran compiler. All experiments were performed on a desktop computer equipped with an AMD 9950X processor and 64GB of RAM. This processor has 16 cores, but only one was utilized in our experiments. In all of our experiments, we took the parameter k controlling the order of the piecewise Chebyshev expansions used to represent Airy phase functions to be 16, and we set precision parameter ϵ to be 10^{-13} . To account for the vagaries of modern computing environments, all reported times were obtained by averaging the cost of each calculation over 100 runs.

For the most part, we measured the accuracy of our method by using it to solve various initial and boundary value problems for second order equations of the form (1). Because the condition numbers of such problems grow with the parameter ω , the accuracy of any solver will deteriorate with increasing ω . In the case of our algorithm, the mechanism by which accuracy is lost is well understood. It computes the Airy phase functions themselves to high precision, but the magnitude of the phase functions increases with the parameter ω , with the consequence that accuracy is lost when the Airy functions are evaluated at large arguments in order to calculate the solutions of the original differential equation. Our algorithm does, however, compute solutions of the original differential equation with accuracy on the order of that predicted by the condition number of the problem being solved.

There is one experiment in which we measured the accuracy of the Airy phase functions produced by our algorithm directly. In the experiment of Subsection 6.3, we constructed Airy phase functions representing associated Legendre functions of various degrees and orders using the algorithm of this paper and then compared them with Airy phase functions produced via another technique in order to show our algorithm computes Airy phase functions with high relative accuracy.

6.1. Initial value problems

In our first experiment, we used the algorithm of this paper to solve initial value problems of the form

$$\begin{cases} y''(t) + \omega^2 q(t)y(t) = 0, & -5 < t < 5, \\ y(0) = 1, \quad y'(0) = 0 \end{cases} \quad (148)$$

for various values of ω and choices of the coefficient $q(t)$. More explicitly, for each of the coefficients

$$q_1(t) = t + t^3, \quad q_2(t) = -1 + (1 - t) \exp(t) \quad \text{and} \quad q_3(t) = t + \frac{\sin(3t)}{3} \quad (149)$$

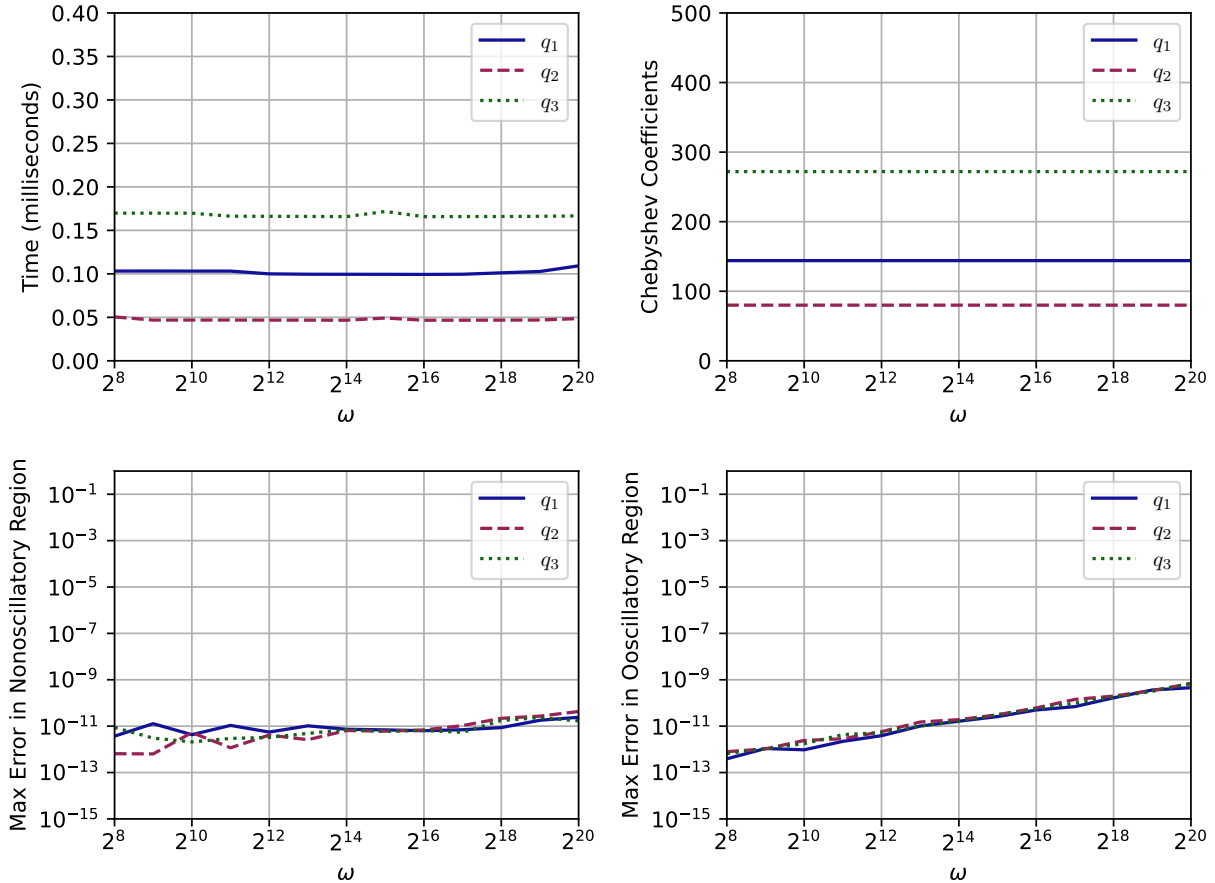


Figure 1: The results of the experiment of Section 6.1 in which a collection of initial value problems were solved using our method. The upper-left plot gives the time, in milliseconds, required to compute each Airy phase function. The plot in the upper right gives the number of coefficients in the piecewise Chebyshev expansions used to represent the Airy phase functions. The plot in the lower left gives the relative accuracy of the obtained solutions of the initial value problems in the nonoscillatory region, and the plot in the lower right reports the absolute accuracy of the obtained solution in the oscillatory region.

and each $\omega = 2^8, 2^9, 2^{10}, \dots, 2^{20}$, we used the algorithm of Section 5 to construct a slowly-varying Airy phase function γ representing the solutions of (148). We then used each of these Airy phase functions to calculate the solution of (148) at a collection of evaluation points in the interval $(-5, 5)$ and compared the obtained values with a reference solution constructed via an adaptive Chebyshev spectral method.

Since each of the coefficients we consider is positive in the interval $(0, 5)$ and negative in the interval $(-5, 0)$, the solutions of (148) oscillate in $(0, 5)$ and are nonoscillatory in $(-5, 0)$. Moreover, the solutions increase very rapidly as the argument t decreases from 0 to -5 . Indeed, in each case we considered, the solutions of (148) were too large to represent via double precision numbers on much of the interval $(-5, 0)$. Accordingly, for each problem, we calculated the largest relative error in the obtained solution at 1,000 equispaced evaluation points in the subinterval $[\tilde{a}, 0]$, where \tilde{a} was chosen so that $\gamma(\tilde{a}) = -100$, and we used this quantity as our measure of the accuracy of the solution of (148) obtained via our method in the nonoscillatory regime. Since

$$\text{Ai}(-100) \approx 4.669498035610554 \times 10^{-291} \quad \text{and} \quad \text{Bi}(-100) \approx 1.070779073708091 \times 10^{289}, \quad (150)$$

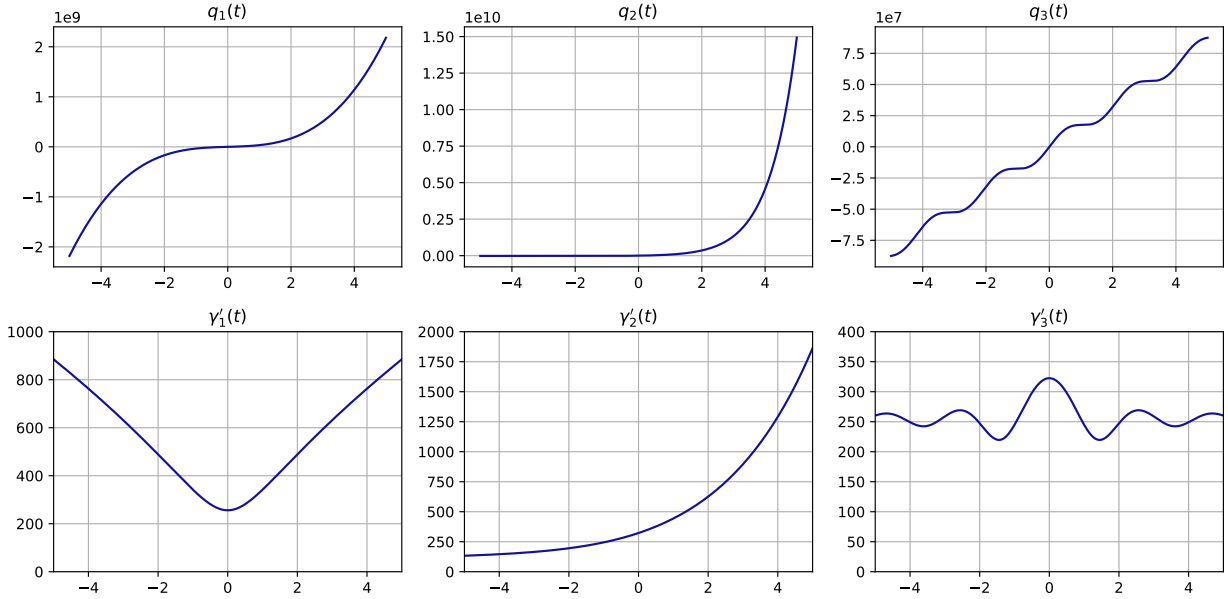


Figure 2: Plots of the coefficients $q_1(t)$, $q_2(t)$ and $q_3(t)$ considered in the experiment of Section 6.1 when the parameter ω is equal to 2^{12} , as well as plots of the derivatives of the corresponding Airy phase functions $\gamma_1'(t)$, $\gamma_2'(t)$ and $\gamma_3'(t)$.

\tilde{a} is close to the point at which the solutions become too large to represent using double precision numbers. We note that the Airy phase functions themselves are computed accurately over the entire interval $[-5, 5]$ via our method. We also used the Airy phase functions to evaluate each solution at 1,000 equispaced evaluation points in the interval $[0, 5]$ and compared those values with the reference solution. Since the solutions are oscillatory here, we measured absolute rather than relative errors at these evaluation points.

Figure 1 gives the results of this experiment. We observe that the running time of our algorithm and the number of Chebyshev coefficients needed to represent the Airy phase functions are both independent of the parameter ω . The errors in the obtained solutions of the initial value problems, on the other hand, increase as ω grows. This is to be expected since the condition number of the each of the problems we considered increases with ω and any numerical algorithm for solving these problems will lose accuracy as ω increases. Plots of the coefficients $q_1(t)$, $q_2(t)$ and $q_3(t)$ and the derivatives of the corresponding Airy phase functions $\gamma_1'(t)$, $\gamma_2'(t)$ and $\gamma_3'(t)$ when $\omega = 2^{12}$ can be found in Figure 2.

6.2. Boundary value problems

In our next experiment, we used the algorithm of this paper to solve the boundary value problem

$$\begin{cases} y''(t) + \omega^2 q(t, \omega) y(t) = 0, & 0 < t < 3, \\ y(0) = 1, & y(3) = 1 \end{cases} \quad (151)$$

for each $\omega = 2^8, 2^9, 2^{10}, \dots, 2^{20}$ and the following choices of q :

$$q_1(t, \omega) = t + t^3, \quad q_2(t, \omega) = \sin(t) + 2 \sin\left(\frac{t}{4}\right)^2 \quad \text{and} \quad q_3(t, \omega) = \frac{t(\cos^2(3t) \sin^2(\omega) + 2)}{t^2 \cos^2(\omega) + 1}. \quad (152)$$

We tested the accuracy of each solution by comparing its value at 1,000 equispaced evaluation points in the solution domain $(0, 3)$ to a reference solution constructed via an adaptive Chebyshev spectral method. Since the solutions are oscillatory in the interval $(0, 3)$, we measured absolute rather than relative errors.

The results are given in Figure 3. As with the experiments of the preceding section, the errors in the obtained solutions increase with ω . But again, this is to be expected since the condition numbers of these boundary value problems grow with ω and a similar loss of accuracy will be experienced by any numerical method. We also note that the errors exhibit greater variability than in the previous experiment. This is due to greater variance in the condition numbers of the boundary value problems we considered. Finally, we observe that the time required to compute the Airy phase function in the case of the coefficient q_3 varies noticeably with ω because, unlike the other coefficients, q_3 depends on ω .

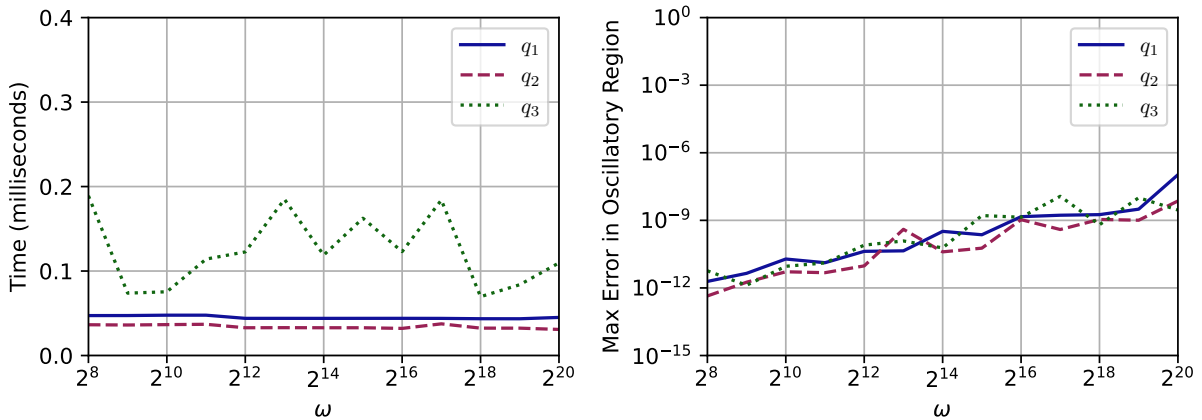


Figure 3: The results of the experiment of Section 6.2 in which a collection of boundary value problems were solved using our method. The plot on the left gives the time, in milliseconds, required to compute each Airy phase function, while the plot on the right gives the accuracy of the obtained solutions of the boundary value problems.

6.3. The associated Legendre differential equation

The standard solutions of the associated Legendre differential equation

$$(1 - t^2)y''(t) - 2ty'(t) + \left(\nu(\nu + 1) - \frac{\mu^2}{1 - t^2} \right) y(t) = 0 \quad (153)$$

on the interval $(-1, 1)$ are the Ferrers functions of the first and second kinds P_ν^μ and Q_ν^μ . Definitions of them can be found in Section 14.3 of [7] or Section 5.15 of [12]. Equation (153) has singular points at ± 1 , and, as long as $\nu > \mu$, it has two turning points in the interval $(-1, 1)$. In order to put this equation into a form suitable for our algorithm, we introduce the change of variables $t = \tanh(x + \xi_\nu^\mu)$, where

$$\xi_\nu^\mu = \operatorname{arccosh} \left(\frac{\sqrt{\nu(\nu + 1)}}{\mu} \right), \quad (154)$$

which yields the new equation

$$y''(x) + (-\mu^2 + \nu(\nu + 1) \operatorname{sech}^2(x - \xi_\nu^\mu)) y(x) = 0. \quad (155)$$

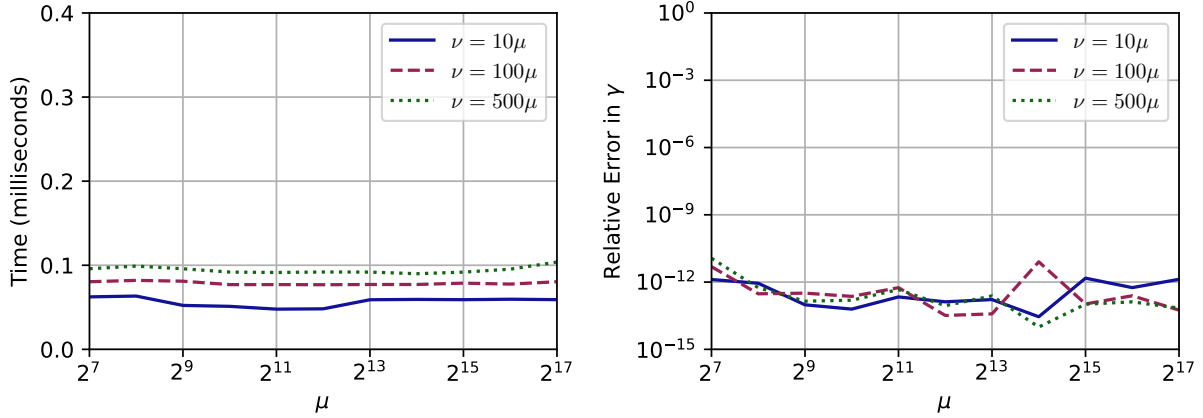


Figure 4: Some of the results of the experiment of Section 6.3, which concern the associated Legendre functions. The plot on the left gives the time required to compute each Airy phase function, while the plot on the right gives the maximum relative error in the Airy phase functions calculated using the method of this paper.

The coefficient in (155) is smooth and while it has two turning points located at 0 and $2\xi_\nu^\mu$, because of the symmetries of the associated Legendre functions, it suffices to consider it on the interval $(-\infty, \xi_\nu^\mu)$ which contains only the turning point at 0.

For each $\mu = 10^7, 10^8, \dots, 10^{17}$ and each $\nu = 10\mu, 100\mu, 500\mu$, we used the algorithm of this paper to construct an Airy phase function γ_ν^μ representing the solutions of (155) over the interval $[a_\nu^\mu, \xi_\nu^\mu]$, where ξ_ν^μ is as in (154) and a_ν^μ is chosen such that $\gamma_\nu^\mu(a_\nu^\mu) = -15$. We note that

$$\text{Bi}(-15) \approx 3.364489547667594 \times 10^{16} \quad \text{and} \quad \text{Ai}(-15) \approx 3.837296156948168 \times 10^{-18}, \quad (156)$$

so that the solutions of (155) are either of very large or very small magnitude at a_ν^μ .

We first tested the accuracy of each γ_ν^μ by using it to evaluate a solution of (155) at 1,000 equispaced points in the solution domain and recording the largest observed relative error. Because the Ferrers functions themselves are normalized such that P_ν^μ and Q_ν^μ become astronomically large on portions of $(-1, 1)$, even for relatively small values of ν and μ , we choose to evaluate a different solution instead. More explicitly, we considered the solution

$$F_\nu^\mu(x) = \tilde{Q}_\nu^\mu(x) + i\tilde{P}_\nu^\mu(x), \quad (157)$$

where \tilde{Q}_ν^μ and \tilde{P}_ν^μ are given via the formulas

$$\tilde{Q}_\nu^\mu(x) = \sqrt{\frac{2}{\pi} \frac{\Gamma(1+\nu+\mu)}{\Gamma(1+\nu-\mu)}} Q_\nu^{-\mu}(\tanh(x)) \quad \text{and} \quad \tilde{P}_\nu^\mu(x) = \sqrt{\frac{\pi}{2} \frac{\Gamma(1+\nu+\mu)}{\Gamma(1+\nu-\mu)}} P_\nu^\mu(\tanh(x)). \quad (158)$$

The Wronskian of this pair is 1, which ensures that F_ν^μ is normalized in a reasonable way. Moreover, \tilde{P}_ν^μ and \tilde{Q}_ν^μ determine a slowly-varying trigonometric phase function α_ν^μ for (155) through the relations

$$\tilde{Q}_\nu^\mu(x) = \frac{\cos(\alpha_\nu^\mu(x))}{\sqrt{\frac{d}{dx}\alpha_\nu^\mu(x)}}, \quad \tilde{P}_\nu^\mu(x) = \frac{\sin(\alpha_\nu^\mu(x))}{\sqrt{\frac{d}{dx}\alpha_\nu^\mu(x)}} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \alpha_\nu^\mu(x) = 0 \quad (159)$$

It is beyond the scope of this paper to show that α_ν^μ is, in fact, slowly varying, but we refer the interested reader to [4]. Since the logarithmic derivative of F_ν^μ is the derivative of α_ν^μ , its condition

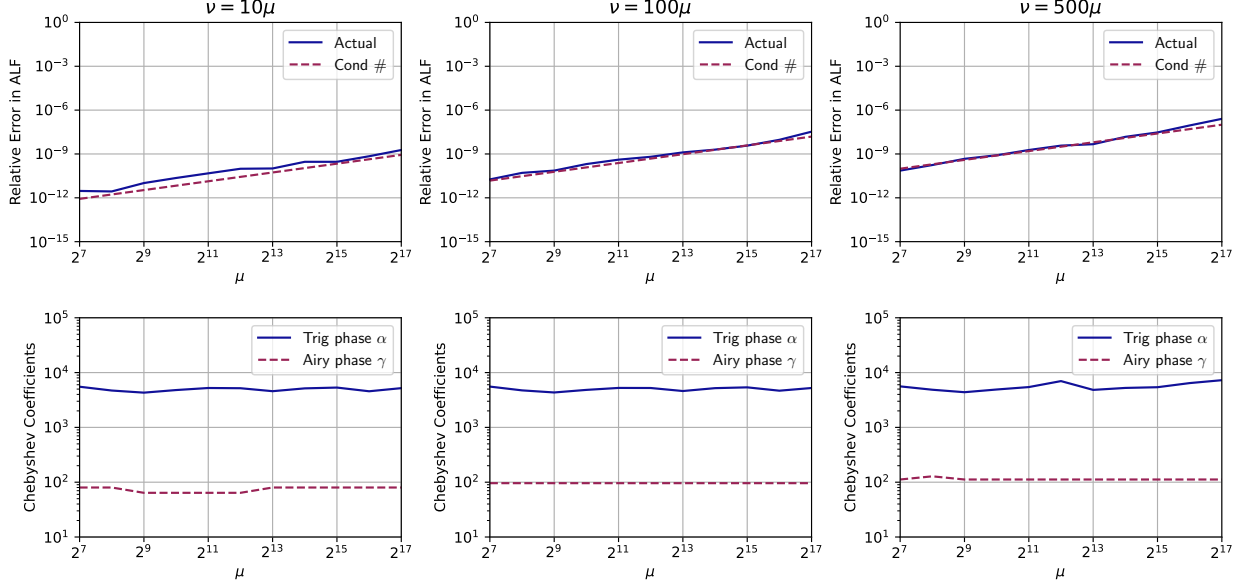


Figure 5: Some of the results of the experiment of Section 6.3, which concerns the associated Legendre functions. The plots on the top row compare the relative error in the evaluation of the function F_ν^μ defined via (157) with the accuracy predicted by its condition number of evaluation. The plots on the bottom row report the number of coefficients in the piecewise Chebyshev expansions of the Airy phase functions γ_ν^μ and the trigonometric phase functions α_ν^μ .

number of evaluation is slowly varying.

Next, for each pair of values of ν and μ considered, we used an alternate approach to construct a second Airy phase function $\tilde{\gamma}_\nu^\mu$ and measured the relative accuracy of γ_ν^μ by comparing the two phase functions at 1,000 equispaced points on the interval $[a_\nu^\mu, \xi_\nu^\mu]$. To construct $\tilde{\gamma}_\nu^\mu$, we first used the algorithm of [15] to calculate the slowly-varying trigonometric phase function α_ν^μ defined via (159) and then composed it with the inverse of a slowly-varying trigonometric phase function α_{ai} for Airy's equation; that is, we let $\tilde{\gamma}_\nu^\mu(t) = \alpha_{\text{ai}}^{-1}(\alpha_\nu^\mu(t))$. The phase function α_{ai} is determined via the requirements

$$\text{Bi}(x) = \frac{\cos(\alpha_{\text{ai}}(x))}{\sqrt{\alpha'_{\text{ai}}(x)}}, \quad \text{Ai}(x) = \frac{\sin(\alpha_{\text{ai}}(x))}{\sqrt{\alpha'_{\text{ai}}(x)}} \quad \text{and} \quad \lim_{x \rightarrow -\infty} \alpha_{\text{ai}}(x) = 0. \quad (160)$$

That the composition $\tilde{\gamma}_\nu^\mu$ is an Airy phase function for (155) follows from the formulas

$$\begin{aligned} \frac{\text{Bi}(\tilde{\gamma}_\nu^\mu(x))}{\sqrt{\frac{d}{dx}\tilde{\gamma}_\nu^\mu(x)}} &= \frac{\cos(\alpha_{\text{ai}}(\tilde{\gamma}_\nu^\mu(x)))}{\sqrt{\alpha'_{\text{ai}}(\tilde{\gamma}_\nu^\mu(x)) \frac{d}{dx}\tilde{\gamma}_\nu^\mu(x)}} = \frac{\cos(\alpha_\nu^\mu(x))}{\sqrt{\frac{d}{dx}\alpha_\nu^\mu(x)}} = \tilde{Q}_\nu^\mu(x) \quad \text{and} \\ \frac{\text{Ai}(\tilde{\gamma}_\nu^\mu(x))}{\sqrt{\frac{d}{dx}\tilde{\gamma}_\nu^\mu(x)}} &= \frac{\sin(\alpha_{\text{ai}}(\tilde{\gamma}_\nu^\mu(x)))}{\sqrt{\alpha'_{\text{ai}}(\tilde{\gamma}_\nu^\mu(x)) \frac{d}{dx}\tilde{\gamma}_\nu^\mu(x)}} = \frac{\sin(\alpha_\nu^\mu(x))}{\sqrt{\frac{d}{dx}\alpha_\nu^\mu(x)}} = \tilde{P}_\nu^\mu(x). \end{aligned} \quad (161)$$

The results of this experiment are given in Figures 4 and 5. We note that, in addition to the maximum relative error observed while evaluating F_ν^μ using our algorithm, the plots on the top row of Figure 5 report the maximum relative accuracy that is expected given the condition number of

evaluation of F_ν^μ , which is, of course,

$$\kappa_\nu^\mu = \max_{x_1, \dots, x_{1000}} \left| x_j \frac{\frac{d}{dx} F_\nu^\mu(x_j)}{F_\nu^\mu(x_j)} \right| \epsilon_0, \quad (162)$$

where $\epsilon_0 \approx 2.220446049250313 \times 10^{-16}$ is machine zero for the IEEE double precision number system. The plots on the bottom row of Figure 5 compare the number of coefficients in the piecewise expansions of the trigonometric phase functions α_ν^μ and of the Airy phase functions γ_ν^μ . While both functions are represented at a cost which is independent of μ , the trigonometric phase function is several orders of magnitude more expensive to represent than the Airy phase function. Figure 6, which contains plots of the derivatives of the trigonometric phase function α_ν^μ and of the Airy phase function γ_ν^μ in the case $\nu = 2560$ and $\mu = 256$, makes clear why this is the case. In particular, it shows that the derivative of α_ν^μ exhibits complicated behavior near the turning point, while γ_ν^μ is extremely benign throughout the interval.

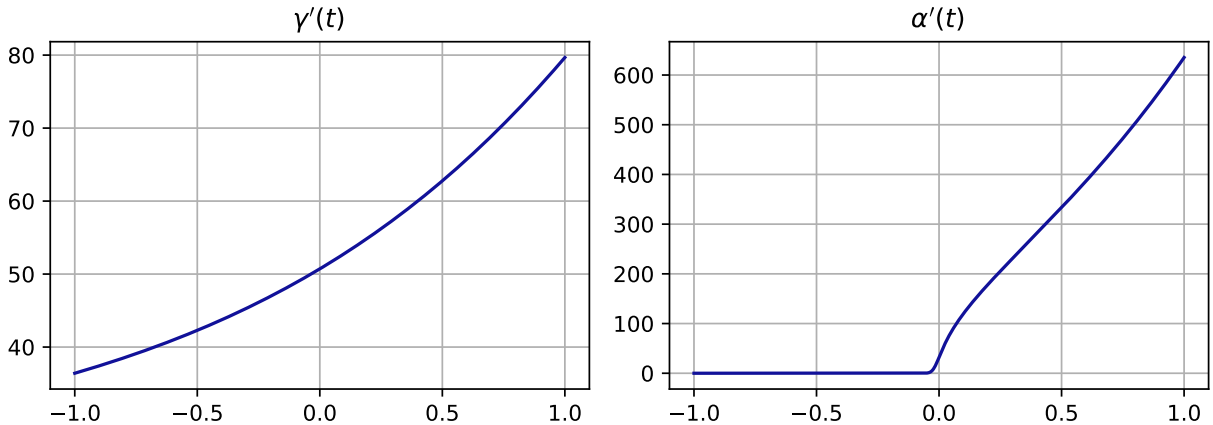


Figure 6: Graphs of the derivatives of the Airy phase function γ_ν^μ (left) and the trigonometric phase α_ν^μ (right) over the interval $[-1, 1]$ in the case $\mu = 256$ and $\nu = 2560$. The derivative of the trigonometric phase function is significantly more expensive to represent via polynomial expansions than the Airy phase function, which is extremely benign.

7. Conclusion

We have given a proof of the existence of slowly-varying Airy phase functions and described a numerical method for rapidly computing them. Using our algorithm, a large class of second order linear ordinary differential equations of the form (1) can be solved to high accuracy in time independent of the parameter ω . This class includes many differential equations defining widely-used special functions, as well as many equations with applications in physics and chemistry.

With some modification, our numerical method extends to the case of equations of the form

$$y''(t) + \omega^2 t^\sigma q(t)y(t) = 0, \quad (163)$$

where $q(t) \sim 1$ as $t \rightarrow 0$, $\sigma > -2$ and ω is large. However, the analysis of this paper fails when σ is not one of the special values $-1, 0$ or 1 . The authors will discuss the necessary modifications to our numerical algorithm and an alternative approach to the procedure of Section 3.1 in a future work.

In the experiment of Subsection 6.3, a second procedure for constructing slowly-varying Airy phase functions was introduced. Namely, the inverse of a trigonometric phase function for Airy's equation

was composed with a trigonometric phase function for the associated Legendre equation in order to form a slowly-varying Airy phase function for the associated Legendre equation. Methods of this type might be useful for various numerical computations as well as for the derivation of asymptotic approximations for the solutions of ordinary differential equations, and this approach warrants further investigation.

Finally, we note that the efficient representation of solutions of second order linear ordinary differential equations via generalized phase functions has many applications to the rapid evaluation of special functions and their zeros, and to the rapid application of the related Sturm-Liouville transforms. The authors also plan to explore these topics in future works.

8. Acknowledgments

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References

- [1] F. J. Agocs, W. J. Handley, A. N. Lasenby, and M. P. Hobson. Efficient method for solving highly oscillatory ordinary differential equations with applications to physical systems. *Phys. Rev. Res.*, 2:013030, Jan 2020.
- [2] Harry Bateman and A. Erdélyi. *Higher Transcendental Functions*, volume I. McGraw-Hill, New York, New York, 1953.
- [3] James Bremer. On the numerical solution of second order differential equations in the high-frequency regime. *Applied and Computational Harmonic Analysis*, 44:312–349, 2018.
- [4] James Bremer. Phase function methods for second order linear ordinary differential equations with turning points. *Applied and Computational Harmonic Analysis*, 65:137–169, 2023.
- [5] P. Ciarlet. *Linear and Nonlinear Functional Analysis with Applications*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.
- [6] R.C. Davidson and Q. Hong. *Physics of intense charged particle beams in high energy accelerators*. World Scientific, Singapore, 2001.
- [7] *NIST Digital Library of Mathematical Functions*. <https://dlmf.nist.gov/>, Release 1.2.2 of 2024-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
- [8] R. D. Hazeltine and J. D. Meiss. *Plasma confinement*. Courier Corporation, North Chelmsford, Massachusetts, 2003.
- [9] L.V. Kantorovich. Functional analysis and applied mathematics. *Uspehi Matematicheskii Nauk*, 3:89–185, 1948.
- [10] E.E. Kummer. De generali quadam aequatione differentiali tertii ordinis. *Progr. Evang. Königl. Stadtgymnasium Liegnitz*, 1834.
- [11] Jerome Martin and Dominik Schwarz. WKB approximation for inflationary cosmological perturbations. *Physical Review D*, 67, 10 2002.

- [12] Frank W.J. Olver. *Asymptotics and Special Functions*. A.K. Peters, Wellesley, Massachusetts, 1997.
- [13] G M Pritula, E. V. Petrenko, and O. V. Usatenko. Adiabatic dynamics of one-dimensional classical Hamiltonian dissipative systems. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 382(8):548–553, feb 2018.
- [14] Zewen Shen and Kirill Serkh. On polynomial interpolation in the monomial basis, arXiv:2212.10519, 2025.
- [15] Tara Stojimirovic and James Bremer. An accelerated frequency-independent solver for oscillatory differential equations, arXiv:2409.18487, 2024.