On the numerical solution of second order ordinary differential equations in the high-frequency regime

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Abstract

We describe an algorithm for the numerical solution of second order linear ordinary differential equations in the high-frequency regime. It is based on the recent observation that solutions of equations of this type can be accurately represented using nonoscillatory phase functions. Unlike standard solvers for ordinary differential equations, the running time of our algorithm is independent of the frequency of oscillation of the solutions. We illustrate this and other properties of the method with several numerical experiments.

Keywords: Ordinary differential equations, fast algorithms, phase functions, special functions, Bessel's equation

1. Introduction

Second order linear differential equations of the form

$$y''(t) + \lambda^2 q(t) y(t) = 0 \quad \text{for all} \quad a \leqslant t \leqslant b \tag{1}$$

are ubiquitous in analysis and mathematical physics. As a consequence, much attention has been devoted to the development of numerical algorithms for their solution and, in most regimes, fast and accurate methods are available.

However, when q is positive and λ is real-valued and large, the solutions of (1) are highly oscillatory (this is a consequence of the Sturm comparison theorem) and standard solvers for ordinary differential equations (for instance, Runge-Kutta schemes and spectral methods) suffer. Specifically, their running times grow linearly with the parameter λ, which makes them prohibitively expensive when λ is large.

Because of the poor performance of standard solvers, asymptotic methods are often used in this regime. In some instances, they allow for the accurate evaluation of solutions of equation of the form (1) using a number of operations which is independent of the parameter λ. For example, [3] presents an $\mathcal{O}(1)$ algorithm for calculating Legendre polynomials of arbitrary order using a combination of direct evaluation and asymptotic formulas; it achieves near machine precision accuracy and serves as the basis for a fast algorithm (also presented in [3]) for the construction of Gauss-Legendre quadratures of extremely large orders. In a similar vein, [14] describes a fast algorithm for the computation of Gauss-Legendre and Gauss-Jacobi quadratures which makes use of asymptotic formulas in order to evaluate Jacobi polynomials in $\mathcal{O}(1)$ operations.

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The formulas used in [3] and [14] are particular to the cases they consider, and while the same approach can be applied to other classes of special functions satisfying equations of the form (1), in each case a new, specialized approach must be devised. Indeed, despite the extensive existing literature on the asymptotic approximation of Legendre polynomials, the algorithm of [3] required the development of a novel asymptotic expansion with suitable numerical properties.

Here, we describe an algorithm for the numerical solution of second order linear ordinary differential equations of the form (1) whose running time is independent of the parameter λ. It applies to a large class of second order ordinary differential equations — which includes those defining Bessel functions, Legendre functions of integer and noninteger orders, prolate spheroidal wave functions, the classical orthogonal polynomials, etc.

Our approach proceeds by constructing a nonoscillatory phase function which represents solutions of (1). We say that α is a phase function for (1) if the functions u, v defined by the formulas

$$u(t) = \frac{\cos(\alpha(t))}{|\alpha'(t)|^{1/2}}, \tag{2}$$

and

$$v(t) = \frac{\sin(\alpha(t))}{|\alpha'(t)|^{1/2}} \tag{3}$$

comprise a basis in the space of solutions of (1). Phase functions play a key role in the theories of special functions and global transformations of ordinary differential equations [4, 20, 21, 1], and are the basis of many numerical algorithms (see [24, 11, 16] for representative examples).

It was observed by E.E. Kummer in [19] that α is a phase function for (1) if and only if it satisfies the third order nonlinear differential equation

$$\left(\alpha'(t)\right)^2 = \lambda^2 q(t) - \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 \tag{4}$$

on the interval $[a, b]$. The presence of quotients in (4) is often inconvenient, and we prefer the more tractable equation

$$r''(t) - \frac{1}{4}\left(r'(t)\right)^2 + 4\lambda^2\left(\exp(r(t)) - q(t)\right) = 0 \tag{5}$$

obtained from (4) by letting

$$\alpha'(t) = \lambda \exp\left(\frac{r(t)}{2}\right). \tag{6}$$

Of course, if r is a solution of (5) then a solution α of (4) such that $\alpha(a) = 0$ is given by the formula

$$\alpha(t) = \lambda \int_a^t \exp\left(\frac{r(u)}{2}\right)\, du. \tag{7}$$

We will refer to (4) as Kummer's equation and (5) as the logarithm form of Kummer's equation. The form of these equations and the appearance of λ in them suggests that their solutions will be oscillatory — and most of them are. However, there are several well-known examples of second order ordinary differential equations which admit nonoscillatory phase functions. For example, the function

$$\alpha(t) = \lambda \arccos(t) \tag{8}$$

is a phase function for Chebyshev's equation

$$y''(t) + \left(\frac{2 + t^2 + 4\lambda^2(1-t^2)}{4(1-t^2)^2}\right) y(t) = 0 \quad \text{for all} \quad -1 \leqslant t \leqslant 1. \tag{9}$$

Bessel's equation

$$y''(t) + \left(1 - \frac{\lambda^2 - 1/4}{t^2}\right) y(t) = 0 \quad \text{for all} \quad 0 < t < \infty \tag{10}$$

also admits a nonoscillatory phase function, although it cannot be expressed via elementary functions; see, for instance, [16].

Exact solutions of (4) which are nonoscillatory need not exist in every case. However, [15] and [5] make the observation that when the coefficient q appearing in (1) is nonoscillatory, there exists a nonoscillatory function α such that (2), (3) approximate solutions of (1) with accuracy on the order of $(\mu\lambda)^{-1}\exp(-\mu\lambda)$, where μ is a constant which depends on the coefficient q but not on λ. More specifically, there exists a nonoscillatory function r which satisfies the equation

$$r''(t) - \frac{1}{4}\left(r'(t)\right)^2 + 4\lambda^2 \left(\exp(r(t)) - q(t)\right) = q(t)\nu, \tag{11}$$

where ν is a smooth function such that

$$\|\nu\|_\infty = \mathcal{O}\left(\frac{1}{\mu}\exp(-\mu\lambda)\right). \tag{12}$$

The function α obtained from r via formula (7) is a solution of the nonlinear differential equation

$$\left(\alpha'(t)\right)^2 = \lambda^2 \left(\frac{\nu(t)}{4\lambda^2} + 1\right) q(t) - \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2; \tag{13}$$

this implies that α is a phase function for the equation

$$y''(t) + \lambda^2 \left(1 + \frac{\nu(t)}{4\lambda^2}\right) q(t) y(t) = 0 \quad \text{for all} \quad a \leqslant t \leqslant b. \tag{14}$$

It follows from (14) and (12) that when α is inserted into formulas (2) and (3), the resulting functions approximate solutions of (1) with $\mathcal{O}\left((\mu\lambda)^{-1}\exp(-\mu\lambda)\right)$ accuracy (see Theorem 12 in [5]). The functions r and α are nonoscillatory in the sense that they can be accurately represented using various series expansions (e.g., expansions in Chebyshev polynomials) whose number of terms does not depend on λ. In other words, $\mathcal{O}(1)$ terms are required to represent the solutions of (1) with $\mathcal{O}\left((\mu\lambda)^{-1}\exp(-\mu\lambda)\right)$ accuracy. This is an improvement over superasymptotic and hyperasymptotic expansions (see, for instance, [9, 8]), which represent solutions of (1) with accuracy on the order of $\exp(-\rho\lambda)$ using expansions with $\mathcal{O}(\lambda)$ terms. Theorem 12 of [5], reproduced as Theorem 3 in Section 3.2 of this article, gives a precise statement regarding the existence of nonoscillatory phase functions.

The existence of the nonoscillatory solution r of (11) is established in [5] by assuming that the coefficient q extends to the real line and considering an integral equation related to (11) there (see Sections 3.1 and 3.2 for details). The function q is extended so that the Fourier transform can be used to quantify the notion of "nonoscillatory" function. This method could serve as the basis of a numerical method for the computation of nonoscillatory phase functions. However, in addition to requiring the extension of q, such an approach also requires knowledge of the first two derivatives of q.

In this article, we describe a method for constructing a solution of the logarithm form of Kummer's equation whose difference from the nonoscillatory solution of (11) is on the order of $\exp\left(-\frac{1}{2}\mu\lambda\right)$. It does not require that q be extended beyond the interval $[a, b]$, nor does it take as input the values of the derivatives of q.

Our approach is based on two observations. First, that if

$$q(t) = 1 + \epsilon(t) \tag{15}$$

for all t in an interval of the form $[a, a+\tau]$, where $\tau > 0$ and ϵ is a smooth function of sufficiently small magnitude, then the difference between the solution of the initial value problem

$$\begin{cases} r''(t) - \frac{1}{4}\left(r'(t)\right)^2 + 4\lambda^2\left(\exp(r(t)) - q(t)\right) = 0 \quad \text{for all} \quad a \leqslant t \leqslant b \\ r(a) = r'(a) = 0 \end{cases} \tag{16}$$

and the nonoscillatory solution of (11) is on the order of $\exp\left(-\frac{1}{2}\mu\lambda\right)$ on the interval $[a, b]$. Second, that when the coefficient q is perturbed by a smooth function φ which is of sufficiently small magnitude on the interval $[a, b]$, the changes induced in the restrictions of the nonoscillatory solution of (11) and its derivative to the interval $[a, b]$ are also on the order of $\exp\left(-\frac{1}{2}\mu\lambda\right)$. These observations follow by combining Theorem 4 of Section 3.3 with standard results on the continuity of solutions of ordinary differential equations with respect to perturbation of initial values and coefficients. The role of Theorem 4 is to bound the magnitude of the nonoscillatory solution r of (11) at the point a under the assumption that q is nearly equal to a constant there. Such an estimate is required because of the manner in which the nonoscillatory solution r of (11) is defined.

We exploit these observations as follows. First, we construct a windowed version $\tilde{q}$ of the function q such that

$$\tilde{q}(t) = \begin{cases} 1 + \epsilon(t) & \text{for all} \quad a \leqslant t < t + \tau \\ q(t) & \text{for all} \quad b - \tau < t \leqslant b, \end{cases} \tag{17}$$

where τ is (once again) a small positive real number and $\epsilon(t)$ is a function of small magnitude, and calculate a solution r_1 of the initial value problem

$$\begin{cases} r_1''(t) - \frac{1}{4}\left(r_1'(t)\right)^2 + 4\lambda^2\left(\exp\left(r_1(t)\right) - \tilde{q}(t)\right) = 0 \quad \text{for all} \quad a \leqslant t \leqslant b \\ r_1(a) = r_1'(a) = 0. \end{cases} \tag{18}$$

Next, we obtain a solution r_2 of the problem

$$\begin{cases} r_2''(t) - \frac{1}{4}\left(r_2'(t)\right)^2 + 4\lambda^2\left(\exp\left(r_2(t)\right) - q(t)\right) = 0 \quad \text{for all} \quad a \leqslant t \leqslant b \\ r_2\left(b\right) = r_1\left(b\right) \quad \text{and} \quad r_2'\left(b\right) = r_1\left(b\right). \end{cases} \tag{19}$$

From our first observation, we see that the difference between the solution of (18) and the nonoscillatory function $\tilde{r}$ obtained by applying Theorem 3 to the equation

$$\tilde{r}''(t) - \frac{1}{4}\left(\tilde{r}'(t)\right)^2 + 4\lambda^2\left(\exp(\tilde{r}(t)) - \tilde{q}(t)\right) = 0 \tag{20}$$

is on the order of $\exp\left(-\frac{1}{2}\mu\lambda\right)$. Moreover, according to our second observation, the difference between the function r_1 and the nonoscillatory solution r of

$$r''(t) - \frac{1}{4}\left(r'(t)\right)^2 + 4\lambda^2\left(\exp(r(t)) - q(t)\right) = q(t)\nu(t) \tag{21}$$

whose existence is guaranteed by Theorem 3 is on the order of

$$\exp\left(-\frac{1}{2}\mu\lambda\right) \tag{22}$$

on the interval $[b - \tau, b]$, as is the difference between the derivatives of these two functions. In particular,

$$\left|r_1\left(b\right) - r\left(b\right)\right| + \left|r_1'\left(b\right) - r'\left(b\right)\right| = O\left(\exp\left(-\frac{1}{2}\mu\lambda\right)\right). \tag{23}$$

Together (12), (19), (21), and (23) imply that

$$\left|r_2(t) - r(t)\right| = O\left(\exp\left(-\frac{1}{2}\mu\lambda\right)\right) \tag{24}$$

for all $t \in [a, b]$. That is, the difference between the solution r_2 of the boundary value problem (19) and the nonoscillatory solution r of (11) decays exponentially with λ.

In the high-frequency regime, the difference between r_1 and the nonoscillatory function $\tilde{r}$ is considerably smaller than machine precision, as is the difference between r_2 and the nonoscillatory function r. Consequently, for the purposes of numerical computation, r_1 and r_2 can be regarded as nonoscillatory. In particular, solutions of the boundary value problems (18) and (19) can be obtained via a standard method for the numerical solution of ordinary differential equations, and each of the functions r_1 and r_2 can be approximated to high accuracy by a finite series expansion whose number of terms does not depend on λ. Moreover, the number of operations required to compute these expansions of r_1 and r_2 does not depend on λ.

There is one significant limitation on the accuracy obtained by the algorithm of this paper. When λ is large, the evaluation of the functions u, v defined via the formulas (2) and (3) requires the computation of trigonometric functions of large arguments. There is an inevitable loss of accuracy when these calculations are performed in finite precision arithmetic. Nonetheless, acceptable accuracy is obtained in many cases. For instance, Section 5.3 describes an experiment in which the Bessel function of the first kind of order 10^8 was evaluated to approximately ten digits of accuracy at a large collection of points on the real axis.

We also note that although some accuracy is lost when evaluating (2), (3) in the high-frequency regime, the phase function α produced by the algorithm of this paper is highly accurate. Among other things, it can be used to rapidly calculate the roots of special functions to high precision. This and other applications of nonoscillatory phase functions will be reported at a later date.

The remainder of this paper is organized as follows. Section 2 summarizes a number of mathematical and numerical facts to be used in the rest of the paper. In Section 3, we develop the analytic apparatus used in Section 4 to devise an algorithm for the rapid solutions of second order linear differential equations in the high-frequency regime. Section 5 presents the results of several numerical experiments conducted to assess the performance of the algorithm of Section 4.

2. Analytic and numerical preliminaries

2.1. Schwartz functions and tempered distributions

We denote by $C^\infty(\mathbb{R})$ the set of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$. We say that an infinitely differentiable function $\varphi : \mathbb{R} \to \mathbb{C}$ is a Schwartz function if φ and all of its derivatives decay faster than any polynomial. That is, if

$$\sup_{t\in\mathbb{R}} |t^i \varphi^{(j)}(t)| < \infty \tag{25}$$

for all pairs i, j of nonnegative integers. The set of all Schwartz functions is denoted by $S(\mathbb{R})$. We endow it with the topology generated by the family of seminorms

$$\|\varphi\|_k = \sum_{j=0}^{k} \sup_{t\in\mathbb{R}} \left| t^k \varphi^{(j)}(x) \right| \quad k = 0, 1, 2, \ldots, \tag{26}$$

so that a sequence $\{\varphi_n\}$ of functions in $S(\mathbb{R})$ converges to φ in $S(\mathbb{R})$ if and only if

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_k = 0 \quad \text{for all} \quad k = 0, 1, 2, \ldots. \tag{27}$$

The space $S(\mathbb{R})$ clearly contains the set $C_c^\infty(\mathbb{R}^n)$ of compactly supported infinitely differentiable functions. We denote the space of continuous linear functionals on $S(\mathbb{R})$, which are known as tempered distributions, by $S'(\mathbb{R})$.

See, for instance, [17] for a thorough discussion of Schwartz functions and tempered distributions.

2.2. The Fourier transform

We define the Fourier transform of a function $f \in S(\mathbb{R})$ via the formula

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} \exp(-ix\xi) f(x)\ dx. \tag{28}$$

The Fourier transform is an isomorphism $S(\mathbb{R}) \to S(\mathbb{R})$ (meaning that it is a continuous, invertible mapping $S(\mathbb{R}) \to S(\mathbb{R})$ whose inverse is also continuous). The formula

$$\langle \widehat{\omega}, \varphi \rangle = \langle \omega, \widehat{\varphi} \rangle \tag{29}$$

extends the Fourier transform to an isomorphism $S'(\mathbb{R}) \to S'(\mathbb{R})$. The definition (29) coincides with (28) when $f \in L^1(\mathbb{R})$. Moreover, when $f \in L^2(\mathbb{R})$,

$$\widehat{f}(\xi) = \lim_{R\to\infty} \int_{-R}^{R} \exp(-ix\xi) f(x)\ dx. \tag{30}$$

Owing to our choice of convention for the Fourier transform,

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi) \tag{31}$$

and

$$\widehat{f \cdot g}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\xi - \eta)\widehat{g}(\eta)\ d\eta \tag{32}$$

whenever f and g are elements of $L^1(\mathbb{R})$. Moreover,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ix\xi)\widehat{f}(\xi)\ d\xi \tag{33}$$

whenever f and $\widehat{f}$ are elements of $L^1(\mathbb{R})$. The observation that f is an entire function when $\widehat{f}$ is a compactly supported distribution is one consequence of the well-known Paley-Wiener theorem. See [12, 13] for a thorough treatment of the Fourier transform.

2.3. The constant coefficient Helmholtz equation

The following theorem is a special case of a more general one which can be found in [18].

Theorem 1. *Suppose that $f \in S(\mathbb{R})$. If λ is a positive real number, then the function g defined by the formula*

$$g(x) = \frac{1}{2\lambda} \int_{-\infty}^{\infty} \sin(\lambda |x - y|) f(y)\ dy \tag{34}$$

is an infinitely differentiable function,

$$g''(x) + \lambda^2 g(x) = f(x) \quad \textit{for all } x \in \mathbb{R}, \tag{35}$$

and

$$\widehat{g}(\xi) = \frac{\widehat{f}(\xi)}{\lambda^2 - \xi^2}. \tag{36}$$

If λ is complex number with positive imaginary part, then the function h defined by the formula

$$h(x) = \frac{1}{2\lambda i} \int_{-\infty}^{\infty} \exp(2\lambda i |x - y|) f(y)\ dy \tag{37}$$

is an infinitely differentiable function,

$$h''(x) + \lambda^2 h(x) = f(x) \quad \textit{for all } x \in \mathbb{R}, \tag{38}$$

and

$$\widehat{h}(\xi) = \frac{\widehat{f}(\xi)}{\lambda^2 - \xi^2}. \tag{39}$$

We interpret the Fourier transform (36) of g as a tempered distribution defined via principal value integrals; that is to say that for all $\varphi \in S(\mathbb{R})$,

$$\left\langle \frac{\widehat{f}(\xi)}{\lambda^2 - \xi^2}, \varphi \right\rangle = \frac{1}{2\lambda} \left(\lim_{\epsilon \to 0} \int_{|\xi - \lambda| > \epsilon} \frac{\widehat{f}(\xi)\varphi(\xi)}{\lambda - \xi}\, d\xi - \lim_{\epsilon \to 0} \int_{|\xi + \lambda| > \epsilon} \frac{\widehat{f}(\xi)\varphi(\xi)}{\lambda + \xi}\, d\xi \right). \tag{40}$$

The following variant of Theorem 1 can be found in [6].

Theorem 2. *Suppose that f is continuous on the interval $[a, b]$, and that λ is a positive real number. Suppose also that $y : [a, b] \to \mathbb{C}$ is twice continuously differentiable, and that*

$$y''(x) + \lambda^2 y(x) = f(x) \quad \text{for all} \quad a \leqslant x \leqslant b. \tag{41}$$

Then

$$y(x) = y(a) \cos(\lambda(x - a)) + \frac{y'(a)}{\lambda} \sin(\lambda(x - a)) + \frac{1}{\lambda} \int_a^x \sin\left(\lambda\left(x - u\right)\right) f(u)\, du \tag{42}$$

for all $a \leqslant x \leqslant b$.

2.4. Schwarzian derivatives

The Schwarzian derivative of a smooth function $f : \mathbb{R} \to \mathbb{R}$ is

$$\{f, t\} = \frac{f'''(t)}{f'(t)} - \frac{3}{2} \left(\frac{f''(t)}{f'(t)} \right)^2. \tag{43}$$

The Schwarzian derivative of $x(t)$ is related to the Schwarzian derivative of its inverse $t(x)$ through the formula

$$\{x, t\} = -\left(\frac{dx}{dt} \right)^2 \{t, x\}. \tag{44}$$

This identity can be found, for instance, in Section 1.13 of [21].

2.5. The Lambert W function

The Lambert W function or product logarithm is the multiple-valued inverse of the function

$$f(z) = z \exp(z). \tag{45}$$

We follow [7] in using W_0 to denote the branch of W which is real-valued and greater than or equal to -1 on the interval $[-1/e, \infty)$. It is immediate from the definition of W_0 that

$$x \exp(x) \leqslant y \quad \text{if and only if} \quad x \leqslant W_0(y). \tag{46}$$

for all real numbers $y \geqslant -1/e$.

2.6. Chebyshev polynomials and interpolation

For each nonnegative integer m, we refer to the collection of points

$$\rho_j = \cos\left(\frac{\pi j}{m}\right), \quad j = 0, 1, \ldots, m \tag{47}$$

as the $(m+1)$-point Chebyshev grid on the interval $[-1,1]$, and we call individal elements of this set Chebyshev nodes or points.

Suppose that $f : [-1,1] \to \mathbb{R}$ is a continuous function. For each integer m, there exists a unique polynomial of degree m which agrees with f on the $(m+1)$-point Chebyshev grid. We refer to this polynomial as the m^{th} order Chebyshev interpolant for f and denote it by $\Psi_m[f]$. In other words, $\Psi_m[f]$ is the polynomial of degree m defined by the requirement that

$$\Psi_m[f](\rho_j) = f(\rho_j) \tag{48}$$

for $j = 0,1,2,\ldots,m$. If f is Lipschitz continuous, then $\Psi_m[f]$ converges to f in $L^\infty([-1,1])$ norm as $m \to \infty$. Moreover, in the event that f is analytic on an ellipse with foci ± 1 the sum of whose semiaxis is $\gamma > 1$,

$$\|\Psi_m[f] - f\|_\infty = O\left(m^{-\gamma}\right). \tag{49}$$

Proofs of these and related facts can be found in [25], for instance.

Given the values of f on the $(m+1)$-point Chebyshev grid $\rho_0, \rho_1, \ldots, \rho_m$, the value of $\Psi[f]$ can be calculated at any point x in $[-1,1]$ via the formula

$$\Psi[f](x) = \left(\sum_{j=0}^{m} \frac{(-1)^j f(\rho_j)}{x-\rho_j}\right) \Bigg/ \left(\sum_{j=0}^{m} \frac{(-1)^j}{x-\rho_j}\right). \tag{50}$$

The process of approximating a function f via $\Phi[f]$ is referred to as Chebyshev interpolation and (50) is known as the barycentric interpolation formula for Chebyshev polynomials. The stability of barycentric interpolation is discussed extensively in [25].

Suppose that $f : [-1,1] \to \mathbb{R}$ is continuous function, that m is a positive integer, and that g is the function defined by the formula

$$g(t) = \int_{-1}^{t} \Psi_m[f](u)\, du. \tag{51}$$

If $v = \{v_0, v_1, \ldots, v_m\}$ is the vector defined by the formula

$$v_j = f(\rho_j) \tag{52}$$

and $w = \{w_0, w_1, \ldots, w_m\}$ is the vector defined by the formula

$$w_j = g(\rho_j), \tag{53}$$

then we refer to the $(m+1)\times(m+1)$ matrix S_m such that

$$S_m v = w \tag{54}$$

as the spectral integration matrix of order m (that such a matrix exists is clear since the underlying operation is linear).

The preceding constructions can be easily modified in order to accommodate functions defined on any finite interval $[a,b]$. For instance, the $(m+1)$-point Chebyshev grid on the interval $[a,b]$ is the set

$$\left\{\frac{b-a}{2}\,\rho_j + \frac{a+b}{2} \;:\; j = 0,1\ldots,m.\right\} \tag{55}$$

Remark 1. *The set (47) is the collection of the extreme points of the m^{th} order Chebyshev polynomial T_m. The roots of Chebyshev polynomials are often used as interpolation nodes instead. There are few meaningful differences between these two choices, although (47) includes the endpoints ± 1, which is convenient when solving boundary value problems for ordinary differential equations.*

3. Analytical apparatus

Here we develop the analytic apparatus used in Section 4 to design an algorithm for the numerical solution of second order linear ordinary differential equations of the form (1) whose running time is independent of the parameter λ.

We will assume throughout this section that the coefficient q in (1) extends to a strictly positive function on the entire real line. We do this so that the notation of "nonoscillatory" can be defined using the Fourier transform. Note that the numerical algorithm described in Section 4 does not require that q be extended outside of the interval $[a, b]$ on which (1) is given.

In Section 3.1, we reformulate Kummer's equation as a nonlinear integral equation in preparation for a statement of the main theorem of [5]. This is done in Section 3.2, and several consequences of this result are discussed there. In Section 3.3, we develop a theorem which bounds the restriction of the solution r of the logarithm form of Kummer's equation to the interval $(-\infty, a]$ under the assumption that the coefficient q is nearly equal to 1 there. This result is recorded as Theorem 4. In Section 3.4, we observe that combining this theorem with standard techniques from the theory of ordinary differential equations suffices to establish the two observations which underlie the algorithm of Section 4.

3.1. Integral equation formulation of Kummer's equation

In this section, we reformulate Kummer's equation

$$\left(\alpha'(t)\right)^2 = \lambda^2 q(t) - \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 \tag{56}$$

as a nonlinear integral equation in preparation for the statement of the principal result of [5].

By letting

$$\alpha'(t) = \lambda \exp\left(\frac{r(t)}{2}\right) \tag{57}$$

in (56) we obtain

$$r''(t) - \frac{1}{4}\left(r'(t)\right)^2 + 4\lambda^2\left(\exp(r(t)) - q(t)\right) = 0, \tag{58}$$

which we refer to as the logarithm form of Kummer's equation. Representing the solution r of (58) in the form

$$r(t) = \log(q(t)) + \delta(t) \tag{59}$$

results in the equation

$$\delta''(t) - \frac{1}{2}\frac{q'(t)}{q(t)}\delta'(t) - \frac{1}{4}\left(\delta'(t)\right)^2 + 4\lambda^2 q(t)\left(\exp(\delta(t) - 1)\right) = q(t)p(t), \tag{60}$$

where the function p is defined by the formula

$$p(t) = \frac{1}{q(t)}\left(\frac{5}{4}\left(\frac{q'(t)}{q(t)}\right)^2 - \frac{q''(t)}{q(t)}\right). \tag{61}$$

By expanding the exponential in (60) in a power series and rearranging terms we obtain

$$\delta''(t) - \frac{1}{2}\frac{q'(t)}{q(t)}\delta'(t) + 4\lambda^2 q(t)\delta(t) - \frac{1}{4}\left(\delta'(t)\right)^2 + 4\lambda^2 q(t)\left(\frac{(\delta(t))^2}{2!} + \frac{(\delta(t))^3}{3!} + \cdots\right) = q(t)p(t). \tag{62}$$

The change of variables

$$x(t) = \int_a^t \sqrt{q(u)}\ du \tag{63}$$

transforms (62) into

$$\delta''(x) + 4\lambda^2\delta(x) = S\left[\delta\right](x) + p(x), \tag{64}$$

where S is the nonlinear differential operator defined via the formula

$$S\left[f\right](x) = \frac{(f'(x))^2}{4} - 4\lambda^2 \left(\frac{(f(x))^2}{2!} + \frac{(f(x))^3}{3!} + \cdots \right). \tag{65}$$

We observe that the function $p(t)$ defined in formula (61) is related to the Schwarzian derivative (see Section 2.4) $\{x, t\}$ of the function x defined in (63) via the formula

$$p(t) = -\frac{2}{q(t)}\{x,t\} = -2\left(\frac{dt}{dx}\right)^2 \{x,t\}. \tag{66}$$

From (66) and Formula (44) in Section 2.4, we see that

$$p(x) = 2\{t, x\}; \tag{67}$$

that is the function $p(x)$ is twice the Schwarzian derivative of the inverse of the function $x(t)$.

We also observe that the differential operator appearing on the left-hand side of (64) is the constant coefficient Helmholtz equation. In order to exploit this observation, we define the operator T for functions $f \in S(\mathbb{R})$ via the formula

$$T\left[f\right](x) = \frac{1}{4\lambda}\int_{-\infty}^{\infty} \sin\left(2\lambda\left|x-y\right|\right) f(y)\ dy \quad \text{for all} \quad x \in \mathbb{R}. \tag{68}$$

According to Theorem 1, $T\left[f\right]$ is the unique solution of the ordinary differential equation

$$y''(x) + 4\lambda^2 y(x) = f(x) \tag{69}$$

such that

$$\widehat{T\left[f\right]}(\xi) = \frac{\widehat{f}(\xi)}{4\lambda^2 - \xi^2}. \tag{70}$$

Consequently, introducing the representation

$$\delta(x) = T\left[\sigma\right](x) \tag{71}$$

into (64) results in the nonlinear integral equation

$$\sigma(x) = S\left[T\left[\sigma\right]\right](x) + p(x). \tag{72}$$

3.2. Nonoscillatory solutions of Kummer's equation

Equation (72) does not admit solutions for all functions p. However, according to the following result, which appears as Theorem 12 in [5], if the function p is nonoscillatory then there exists a function ν of small magnitude such that the nonlinear integral equation

$$\sigma(x) = S\left[T\left[\sigma\right]\right] + p(x) + \nu(x) \tag{73}$$

admits a solution σ which is also nonoscillatory.

Theorem 3. *Suppose that $q \in C^\infty(\mathbb{R})$ is strictly positive, that $x(t)$ is defined by the formula*

$$x(t) = \int_0^t \sqrt{q(u)}\ du, \tag{74}$$

and that the function p defined via the formula

$$p(x) = 2\{t, x\} \tag{75}$$

is an element of $S(\mathbb{R})$. *Suppose further that there exist positive real numbers* λ, Γ *and* μ *such that*

$$\lambda \geqslant 4 \max\left\{\frac{1}{\mu}, \Gamma\right\} \tag{76}$$

and

$$|\widehat{p}(\xi)| \leqslant \Gamma \exp\left(-\mu\, |\xi|\right) \quad \text{for all} \;\; \xi \in \mathbb{R}. \tag{77}$$

Then there exist functions σ *and* ν *in* $S(\mathbb{R})$ *such that* σ *is the unique solution of the nonlinear integral equation*

$$\sigma(x) = S\left[T\left[\sigma\right]\right](x) + p(x) + \nu(x), \quad \text{for all} \;\; x \in \mathbb{R}, \tag{78}$$

$$|\widehat{\sigma}(\xi)| \leqslant \frac{3}{2\Gamma} \exp\left(-\mu|\xi|\right) \quad \text{for all} \;\; |\xi| \leqslant \sqrt{2}\lambda, \tag{79}$$

$$\widehat{\sigma}(\xi) = 0 \quad \text{for all} \;\; |\xi| > \sqrt{2}\lambda, \tag{80}$$

and

$$\|\nu\|_\infty \leqslant \frac{\Gamma}{\mu} \exp\left(-\mu\lambda\right). \tag{81}$$

Suppose that σ and ν are the functions obtained by invoking Theorem 3, and that $x(t)$ is the function defined by the formula

$$x(t) = \int_a^t \sqrt{q(u)}\, du. \tag{82}$$

We define δ by the formula

$$\delta(x) = T\left[\sigma\right](x), \tag{83}$$

r by the formula

$$r(t) = \log(q(t)) + \delta(x(t)), \tag{84}$$

and α by the formula

$$\alpha(t) = \lambda \int_a^t \exp\left(\frac{r(u)}{2}\right)\, du. \tag{85}$$

From the discussion in Section 3.1, we conclude that $\delta(x)$ is a solution of the nonlinear differential equation

$$\delta''(x) + 4\lambda^2\delta(x) = S\left[\delta\right](x) + p(x) + \nu(x) \quad \text{for all} \;\; x \in \mathbb{R}, \tag{86}$$

that $\delta(x(t))$ is a solution of the nonlinear differential equation

$$\delta''(t) - \frac{1}{2}\frac{q'(t)}{q(t)}\delta'(t) - \frac{1}{4}\left(\delta'(t)\right)^2 + 4\lambda^2 q(t)\left(\exp(\delta(t)) - 1\right) = q(t)\left(p(t) + \nu(t)\right) \quad \text{for all} \quad t \in \mathbb{R}, \tag{87}$$

that $r(t)$ is a solution of the nonlinear differential equation

$$r''(t) - \frac{1}{4}(r'(t))^2 + 4\lambda^2\left(\exp(r(t)) - q(t)\right) = q(t)\nu(t) \quad \text{for all} \;\; t \in \mathbb{R}, \tag{88}$$

and that α is a solution of the nonlinear differential equation

$$\left(\alpha'(t)\right)^2 = \lambda^2\left(\frac{\nu(t)}{4\lambda^2} + 1\right)q(t) - \frac{1}{2}\frac{\alpha'''(t)}{\alpha'(t)} + \frac{3}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 \quad \text{for all} \;\; t \in \mathbb{R}. \tag{89}$$

From (89), we see that α is a phase function for the second order linear ordinary differential equation

$$y''(t) + \lambda^2 \left(1 + \frac{\nu(t)}{4\lambda^2}\right) q(t)y(t) = 0 \quad \text{for all} \quad a \leqslant t \leqslant b. \tag{90}$$

The following consequence of Theorem 3, which appears as Theorem 14 in [5], bounds the order of magnitude of the difference between solutions of (90) and those of (1).

Corollary 1. *Suppose that the hypotheses of Theorem 3 are satisfied, that σ and ν are the functions obtained by invoking it. Suppose also that α is defined as in (85), and that u, v are the functions defined via the formulas*

$$u(t) = \frac{\cos(\alpha(t))}{\sqrt{\alpha'(t)}} \tag{91}$$

and

$$v(t) = \frac{\sin(\alpha(t))}{\sqrt{\alpha'(t)}}. \tag{92}$$

Then there exist a constant C and a basis $\{\tilde{u}, \tilde{v}\}$ in the space of solutions of (1) such that

$$|u(t) - \tilde{u}(t)| \leqslant \frac{C}{\lambda} \exp\left(-\mu\lambda\right) \quad \textit{for all} \quad a \leqslant t \leqslant b \tag{93}$$

and

$$|v(t) - \tilde{v}(t)| \leqslant \frac{C}{\lambda} \exp\left(-\mu\lambda\right) \quad \textit{for all} \quad a \leqslant t \leqslant b. \tag{94}$$

The constant C depends on the coefficient q appearing in (1), but not on the parameter λ.

3.3. A bound on δ in the event that p is small in magnitude

In this section, we bound the solution δ of the nonlinear differential equation (86) and its derivative under an assumption on the function p appearing in (86). More specifically, we show that when the function p is of sufficiently small magnitude on the interval $(-\infty, a]$ and λ is sufficiently large, the restrictions of δ and δ' to $(-\infty, a]$ are on the order of

$$\exp\left(-\frac{\lambda\mu}{2}\right). \tag{95}$$

This result, whose proof is the purpose of this section, is recorded as Theorem 4.

Deriving bounds on the magnitude of δ and δ' on the infinite interval $(-\infty, a]$ is complicated by the fact that the operator T is defined via convolution with the nonintegrable kernel

$$\frac{1}{4\lambda} \sin\left(2\lambda|x|\right). \tag{96}$$

To sidestep this difficulty, we perturb the parameter λ in the linear differential operator appearing on the left-hand side of Equation (86) by an imaginary constant $i\eta$ of small magnitude. To be more precise, for each sufficiently small $\eta > 0$ we define the operator T_η for functions $f \in S(\mathbb{R})$ via the formula

$$T_\eta\left[f\right](x) = \frac{1}{4\left(\lambda + i\eta\right)} \int_{-\infty}^{\infty} \exp\left(2(\lambda + i\eta)\left|x - y\right|\right) f(y)\, dy. \tag{97}$$

and we let δ_η denote the function defined via

$$\delta_\eta(x) = T_\eta\left[\sigma\right](x). \tag{98}$$

According to Theorem 1, for each $\eta > 0$, δ_η is the unique solution of the equation

$$\delta_\eta''(x) + 4(\lambda + i\eta)^2 \delta_\eta(x) = \sigma(x) \tag{99}$$

such that

$$\widehat{T_\eta\left[\delta_\eta\right]}(\xi) = \frac{\widehat{\sigma}(\xi)}{4(\lambda+i\eta)^2-\xi^2} \quad \text{for all} \quad \xi \in \mathbb{R}. \tag{100}$$

Since σ is a solution of the integral equation (73), it follows from (99) that

$$\delta_\eta''(x) + 4(\lambda+i\eta)^2\delta_\eta(x) = S\left[\delta\right](x) + p(x) + \nu(x). \tag{101}$$

Note that it is indeed $S\left[\delta\right]$ and not $S\left[\delta_\eta\right]$ which appears on the right-hand side of (101). The advantage of (101) over the nonlinear differential equation

$$\delta''(x) + 4\lambda^2\delta(x) = S\left[\delta\right](x) + p(x) + \nu(x) \tag{102}$$

defining δ is that the fundamental solution

$$\frac{1}{4(\lambda+i\eta)i}\exp\left(2(\lambda+i\eta)i|x|\right) \tag{103}$$

of the differential equation

$$y''(x) + 4(\lambda+i\eta)^2 y(x) = 0 \tag{104}$$

associated with the Fourier transform is an element of $L^1\left(\mathbb{R}\right)$.

The proof of Theorem 4 involves four technical lemmas. The first of these, Lemma 1, bounds the functions $|\delta(x)-\delta_\eta(x)|$ and $\left|\delta'(x)-\delta_\eta'(x)\right|$ for all x on the real line. Lemmas 2 and 3 provide inequalities which are used in Lemma 4 to bound the magnitudes of δ_η and δ_η' on the interval $(-\infty,a]$ under assumptions on the magnitude of the function p there. Lemma 4 is established via a standard "continuity" argument. That is, we use the continuity of the functions δ_η and δ_η' to show that the subset of $(-\infty,a]$ on which the relevant bound is satisfied is nonempty, open and closed in the relative topology of $(-\infty,a]$. Since $(-\infty,a]$ is a connected space, it follows that the bound is satisfied on all of $(-\infty,a]$. Lemma 4 is then combined with the bounds on $|\delta-\delta_\eta|$ and $\left|\delta'-\delta_\eta'\right|$ provided by Lemma 1 to establish the principal result of this section, Theorem 4, which bounds the magnitude of δ and δ' on $(-\infty,a]$ under the assumption that the function p is small there.

Lemma 1. *Suppose that $\sigma \in S(\mathbb{R})$, that there exist positive real numbers μ, Γ and λ such that*

$$|\widehat{\sigma}(\xi)| \leqslant \frac{3\Gamma}{2}\exp\left(-\mu\,|\xi|\right) \quad \textit{for all} \quad |\xi| < \sqrt{2}\lambda, \tag{105}$$

and that

$$\widehat{\sigma}(\xi) = 0 \quad \textit{for all} \quad |\xi| \geqslant \sqrt{2}\lambda. \tag{106}$$

Suppose also that η is a positive real number such that

$$2\eta \leqslant \lambda. \tag{107}$$

Suppose further that δ is defined via the formula

$$\delta(x) = T\left[\sigma\right](x) = \frac{1}{4\lambda}\int_{-\infty}^{\infty}\sin\left(2\lambda\,|x-y|\right)\sigma(y)\;dy, \tag{108}$$

and that δ_η is defined via the formula

$$\delta_\eta(x) = T_\eta\left[\sigma\right](x) = \frac{1}{4\left(\lambda+i\eta\right)i}\int_{-\infty}^{\infty}\exp\left(2\left(\lambda+i\eta\right)i\,|x-y|\right)\sigma(y)\;dy. \tag{109}$$

Then

$$\lim_{|x|\to\infty}|\delta_\eta(x)| + \left|\delta_\eta'(x)\right| = 0, \tag{110}$$

$$|\delta(x)| \leqslant \frac{3\Gamma}{4\pi\mu\lambda^2} \quad \textit{for all} \quad x \in \mathbb{R}, \tag{111}$$

$$\left|\delta'(x)\right| \leqslant \frac{3\Gamma}{4\pi\mu^2\lambda^2}, \quad \textit{for all} \;\; x \in \mathbb{R}, \tag{112}$$

$$|\delta(x) - \delta_\eta(x)| \leqslant \frac{3\Gamma\eta}{\pi\mu\lambda^3} \quad \textit{for all} \;\; x \in \mathbb{R}, \tag{113}$$

and

$$\left|\delta'(x) - \delta'_\eta(x)\right| \leqslant \frac{3\Gamma\eta}{\pi\mu^2\lambda^3} \quad \textit{for all} \;\; x \in \mathbb{R}. \tag{114}$$

Proof. We observe that functions

$$\frac{\widehat{\sigma}(\xi)}{4(\lambda + i\eta)^2 - \xi^2} \tag{115}$$

and

$$\frac{i\xi\widehat{\sigma}(\xi)}{4(\lambda + i\eta)^2 - \xi^2}, \tag{116}$$

are elements of $C_c^\infty(\mathbb{R})$. Among other things, this implies that the inverse Fourier transforms of (115) and (116), which are δ_η and δ'_η, respectively, are elements of $S(\mathbb{R})$. The conclusion (110) follows immediately from this observation.

An elementary calculation shows that

$$\left|\frac{1}{4(\lambda + i\eta)^2 - \xi^2} - \frac{1}{4\lambda^2 - \xi^2}\right| = \left|\frac{4\eta}{4\lambda^2 - \xi^2}\right| \sqrt{\frac{4\lambda^2 + \eta^2}{(4\lambda^2 - \xi^2 - 4\eta^2)^2 + 64\eta^2\lambda^2}}. \tag{117}$$

We observe that

$$\left|\frac{4\eta}{4\lambda^2 - \xi^2}\right| \leqslant \frac{2\eta}{\lambda^2} \tag{118}$$

for all $\eta > 0$ and $|\xi| \leqslant \sqrt{2}\lambda$. Moreover,

$$4\lambda^2 - 4\eta^2 - \xi^2 \geqslant 2\lambda^2 - 4\eta^2 \geqslant 0 \tag{119}$$

for all $|\xi| \leqslant \sqrt{2}\lambda$ and $\lambda \geqslant 2\eta$. It follows from (119) that

$$\begin{aligned} \frac{4\lambda^2 + \eta^2}{(4\lambda^2 - \xi^2 - 4\eta^2)^2 + 64\eta^2\lambda^2} &\leqslant \frac{4\lambda^2 + \eta^2}{(2\lambda^2 - 4\eta^2)^2 + 64\eta^2\lambda^2} \\ &= \frac{4\lambda^2 + \eta^2}{4\lambda^4 + 48\lambda^2\eta^2 + 16\eta^4} \\ &\leqslant \frac{1}{\lambda^2}\frac{4\lambda^2 + \eta^2}{4\lambda^2 + 48\eta^2} \\ &\leqslant \frac{1}{\lambda^2} \end{aligned} \tag{120}$$

for all $0 < 2\eta \leqslant \lambda$, and $|\xi| \leqslant \sqrt{2}\lambda$. By inserting (120) and (118) into (117),we conclude that

$$\left|\frac{1}{4(\lambda + i\eta)^2 - \xi^2} - \frac{1}{4\lambda^2 - \xi^2}\right| \leqslant \frac{2\eta}{\lambda^3} \tag{121}$$

for all $0 < 2\eta \leqslant \lambda$ and $|\xi| < \sqrt{2}\lambda$. From Theorem 1 and (108) we see that

$$\|\delta\|_\infty \leqslant \frac{1}{2\pi}\left\|\widehat{\delta}\right\|_1 \leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\widehat{\sigma}(\xi)}{4\lambda^2 - \xi^2}\right| \, d\xi, \tag{122}$$

and

$$\|\delta'\|_\infty \leqslant \frac{1}{2\pi}\left\|\widehat{\delta'}\right\|_1 \leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{i\xi\widehat{\sigma}(\xi)}{4\lambda^2-\xi^2}\right| d\xi. \tag{123}$$

We insert (105) and (106) into (122) to obtain

$$\|\delta\|_\infty \leqslant \frac{3\Gamma}{4\pi}\int_{-\sqrt{2}\lambda}^{\sqrt{2}\lambda}\left|\frac{1}{4\lambda^2-\xi^2}\right|\exp\left(-\mu|\xi|\right)\,d\xi \leqslant \frac{3\Gamma}{8\pi\lambda^2}\int_{-\sqrt{2}\lambda}^{\sqrt{2}\lambda}\exp\left(-\mu|\xi|\right)\,d\xi \leqslant \frac{3\Gamma}{4\pi\mu\lambda^2}, \tag{124}$$

which is (111). By inserting (105) and (106) into (123) we obtain

$$\|\delta'\|_\infty \leqslant \frac{3\Gamma}{4\pi}\int_{-\sqrt{2}\lambda}^{\sqrt{2}\lambda}\left|\frac{i\xi}{4\lambda^2-\xi^2}\right|\exp\left(-\mu|\xi|\right)\,d\xi \leqslant \frac{3\Gamma}{8\pi\lambda^2}\int_{-\sqrt{2}\lambda}^{\sqrt{2}\lambda}|\xi|\exp\left(-\mu|\xi|\right)\,d\xi \leqslant \frac{3\Gamma}{4\pi\mu^2\lambda^2}, \tag{125}$$

which is (112). Next, we observe that

$$\|\delta-\delta_\eta\|_\infty \leqslant \frac{1}{2\pi}\left\|\widehat{\delta}-\widehat{\delta_\eta}\right\|_1 \leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\widehat{\sigma}(\xi)}{4\lambda^2-\xi^2}-\frac{\widehat{\sigma}(\xi)}{4(\lambda+i\eta)^2-\xi^2}\right| d\xi, \tag{126}$$

and that

$$\|\delta'-\delta'_\eta\|_\infty \leqslant \frac{1}{2\pi}\left\|i\xi\widehat{\delta}-i\xi\widehat{\delta_\eta}\right\|_1 \leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{i\xi\widehat{\sigma}(\xi)}{4\lambda^2-\xi^2}-\frac{i\xi\widehat{\sigma}(\xi)}{4(\lambda+i\eta)^2-\xi^2}\right| d\xi. \tag{127}$$

We insert (105), (106) and (121) into (126) to establish that

$$\|\delta-\delta_\eta\|_\infty \leqslant \frac{3\Gamma\eta}{2\pi\lambda^3}\int_{-\sqrt{2}\lambda}^{\sqrt{2}\lambda}\exp\left(-\mu|\xi|\right)\,d\xi \leqslant \frac{3\Gamma\eta}{\pi\mu\lambda^3}, \tag{128}$$

for all $0<\eta<2\lambda$, which is the conclusion (113). Finally, we combine (105), (106) and (121) with (127) to conclude that

$$\|\delta'-\delta'_\eta\|_\infty \leqslant \frac{3\Gamma\eta}{2\pi\lambda^3}\int_{-\sqrt{2}\lambda}^{\sqrt{2}\lambda}|\xi|\exp\left(-\mu|\xi|\right)\,d\xi \leqslant \frac{3\Gamma\eta}{\pi\mu^2\lambda^3}, \tag{129}$$

for all $0<2\eta<\lambda$, which establishes (114). □

The following technical lemma, which will be used in the proof of Lemma 4, bounds the magnitude of $S[\delta]$, where S is the nonlinear differential operator defined in (65), in terms of the solution δ_η of the complexified equation and its derivative.

Lemma 2. *Suppose that the hypotheses of Lemma 1 are satisfied, and that*

$$\lambda \geqslant 4\max\left\{\Gamma, \frac{1}{\mu}\right\}. \tag{130}$$

Suppose further that S is the nonlinear differential operator defined via (65). Then

$$|S[\delta](x)| \leqslant \frac{\eta^2}{68}+\frac{\left|\delta'_\eta(x)\right|^2}{2}+\frac{102\lambda^2}{25}\left|\delta_\eta(x)\right|^2 \quad \textit{for all} \quad x\in\mathbb{R}. \tag{131}$$

Proof. We define the function τ via the formula

$$\tau(x) = \delta(x)-\delta_\eta(x) \tag{132}$$

so that $\delta=\delta_\eta+\tau_\eta$. We invoke Lemma 1 and exploit the assumption (130) to obtain

$$|\tau(x)| \leqslant \frac{3\eta}{16\pi\lambda}, \quad \text{for all} \quad x\in\mathbb{R} \tag{133}$$

$$\left|\tau'(x)\right| \leqslant \frac{3\eta}{64\pi} \quad \text{for all} \quad x\in\mathbb{R}. \tag{134}$$

and

$$|\delta(x)| \leqslant \frac{3}{64\pi} \quad \text{for all} \quad x \in \mathbb{R}. \tag{135}$$

From the definition (65) of S and the triangle inequality we conclude that

$$|S[\delta](x)| \leqslant \frac{\left|(\delta'(x))^2\right|}{4} + 4\lambda^2 \left|\exp(\delta(x)) - \delta(x) - 1\right| \tag{136}$$

for all $x \in \mathbb{R}$. We insert the inequality

$$|\exp(x) - x - 1| \leqslant \frac{|x|^2}{2} \exp(|x|) \quad \text{for all} \quad x \in \mathbb{R} \tag{137}$$

into (136) to obtain

$$|S[\delta](x)| \leqslant \frac{|\delta'(x)|^2}{4} + 2\lambda^2 \exp(|\delta(x)|) |\delta(x)|^2 \quad \text{for all} \quad x \in \mathbb{R}. \tag{138}$$

From (135) we obtain

$$\exp(|\delta(x)|) \leqslant \exp\left(\frac{3}{64\pi}\right) \leqslant \frac{102}{100} \quad \text{for all} \quad x \in \mathbb{R}. \tag{139}$$

We insert (139) into (138) to see that

$$|S[\delta](x)| \leqslant \frac{|\delta'(x)|^2}{4} + \frac{204\lambda^2}{100} |\delta(x)|^2 \quad \text{for all} \quad x \in \mathbb{R}. \tag{140}$$

We combine (132) with (140) and the fact that

$$(|x+y|)^2 \leqslant (|x| + |y|)^2 \leqslant 2\left(|x|^2 + |y|^2\right) \quad \text{for all} \quad x, y \in \mathbb{R} \tag{141}$$

to conclude that

$$|S[\delta](x)| \leqslant \frac{\left|\delta_\eta'(x)\right|^2 + |\tau'(x)|^2}{2} + \frac{408\lambda^2}{100}\left(|\delta_\eta(x)|^2 + |\tau(x)|^2\right) \tag{142}$$

for all $x \in \mathbb{R}$. Next we insert (133) and (134) into (142) to see that

$$|S[\delta](x)| \leqslant \frac{\eta^2}{68} + \frac{\left|\delta_\eta'(x)\right|^2}{2} + \frac{102\lambda^2}{25} |\delta_\eta(x)|^2 \tag{143}$$

for all $x \in \mathbb{R}$, which is the conclusion of the theorem. □

Our third technical lemma bounds the magnitude of the Fourier transform of the product of σ with a decaying exponential function at the points $\pm 2\lambda$.

Lemma 3. *Suppose that the hypotheses of Lemma 1 are satisfied, and that x is an arbitrary real number. Then*

$$\left|\int_{-\infty}^{\infty} \exp(\pm 2\lambda i y) \exp(-2\eta |x - y|) \sigma(y)\, dy\right| \leqslant \frac{3\Gamma}{2} \exp\left(-\frac{\lambda\mu}{2}\right). \tag{144}$$

Proof. For any $x \in \mathbb{R}$, we define the function g_x via the formula

$$g_x(y) = \exp(-2\eta |x - y|). \tag{145}$$

We observe that

$$\widehat{g_x}(\xi) = \frac{4\eta \exp(-ix\xi)}{4\eta^2 + \xi^2}. \tag{146}$$

Consequently,

$$\widehat{\sigma \cdot g_x}(\pm 2\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\sigma}(\pm 2\lambda - \xi)\ \widehat{g_x}(\xi)\ d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{\sigma}(\pm 2\lambda - \xi) \frac{4\eta \exp(-ix\xi)}{4\eta^2 + \xi^2}\ d\xi. \tag{147}$$

We insert (105) and (106) into (147) to conclude that

$$\begin{aligned} \left|\widehat{g_x \cdot \sigma}(\pm 2\lambda)\right| &\leqslant \frac{3\Gamma}{4\pi} \int_{-\sqrt{2}\lambda}^{\sqrt{2}\lambda} \exp\left(-\left|\pm 2\lambda - \xi\right| \mu\right) \frac{4\eta}{4\eta^2 + \xi^2} d\xi \\ &\leqslant \frac{3\Gamma}{4\pi} \exp\left(-\left(2\lambda - \sqrt{2}\lambda\right)\mu\right) \int_{-\infty}^{\infty} \frac{4\eta}{4\eta^2 + \xi^2} d\xi \\ &\leqslant \frac{3\Gamma}{2} \exp\left(-\frac{\lambda\mu}{2}\right). \end{aligned} \tag{148}$$

In the third line of (148) we used the (easily verifiable) fact that

$$\int_{-\infty}^{\infty} \frac{4\eta}{4\eta^2 + \xi^2} d\xi = 2\pi \quad \text{for all} \quad \eta > 0. \tag{149}$$

□

We now combine Lemmas 1, 2 and 3 to develop a bound on the solution δ_η of the complexified equation

$$\delta_\eta''(x) + 4(\lambda + i\eta)^2 \delta_\eta(x) = \sigma(x) = S\left[\delta\right](x) + p(x) + \nu(x) \tag{150}$$

and its derivative on the interval $(-\infty, a]$ under the assumption that the function p is of small magnitude on that interval. The proof proceeds via a standard "continuity" argument.

Lemma 4. *Suppose that the hypotheses of Theorem 3 are satisfied, and that σ and ν are the functions obtained by invoking it. Suppose further that $\eta > 0$ and a are real numbers, that*

$$\lambda \geqslant \max\left\{\frac{4}{\mu}, 4\Gamma, 2\eta, \frac{2}{\mu}\log\left(\frac{24\Gamma}{\eta}\right), \frac{1}{\mu}\log\left(\frac{16\Gamma}{\mu\eta^2}\right)\right\}, \tag{151}$$

and that

$$|p(x)| \leqslant \frac{\eta^2}{16} \quad \textit{for all} \quad x \leqslant a. \tag{152}$$

Suppose also that δ_η is defined via the formula

$$\delta_\eta(x) = T_\eta\left[\sigma\right](x). \tag{153}$$

Then

$$|\delta_\eta(x)| \leqslant \frac{\eta}{4\lambda} \tag{154}$$

and

$$\left|\delta_\eta'(x)\right| \leqslant \frac{\eta}{4} \tag{155}$$

for all $x \leqslant a$.

Proof. From the definition (97) of T_η and (153) we see that

$$\begin{aligned} \delta_\eta(x) &= \frac{1}{4(\lambda + i\eta)i} \int_{-\infty}^{\infty} \exp\left(2(\lambda + i\eta)i\left|x - y\right|\right)\ \sigma(y)\ dy \\ &= \frac{\exp(2\lambda i x)}{4(\lambda + i\eta)i} \int_{-\infty}^{x} \exp\left(-2\eta\left|x - y\right|\right) \exp(-2\lambda i y)\ \sigma(y)\ dy \\ &+ \frac{\exp(-2\lambda i x)}{4(\lambda + i\eta)i} \int_{x}^{\infty} \exp(-2\eta\left|x - y\right|) \exp(2\lambda i y)\ \sigma(y)\ dy \end{aligned} \tag{156}$$

for all $x \in \mathbb{R}$. By differentiating (156) we conclude that

$$\begin{aligned} \delta_\eta'(x) &= \frac{1}{2}\int_{-\infty}^{\infty} \exp\left(2(\lambda + i\eta)i\left|x-y\right|\right)\ \text{sign}(x-y)\ \sigma(y)\ dy \\ &= \frac{\exp(2\lambda i x)}{2}\int_{-\infty}^{x} \exp\left(-2\eta\left|x-y\right|\right)\exp(-2\lambda i y)\ \sigma(y)\ dy \\ &- \frac{\exp(-2\lambda i x)}{2}\int_{x}^{\infty} \exp(-2\eta\left|x-y\right|)\exp(2\lambda i y)\ \sigma(y)\ dy \end{aligned} \tag{157}$$

for all $x \in \mathbb{R}$. We observe that

$$\begin{aligned} \int_{x}^{\infty} \exp(-2\eta\left|x-y\right|)\exp(2\lambda i y)\ \sigma(y)\ dy &= \int_{-\infty}^{\infty} \exp(-2\eta\left|x-y\right|)\exp(2\lambda i y)\ \sigma(y)\ dy \\ &- \int_{-\infty}^{x} \exp(-2\eta\left|x-y\right|)\exp(2\lambda i y)\ \sigma(y)\ dy \end{aligned} \tag{158}$$

for all $x \in \mathbb{R}$. By inserting (158) into (156), taking absolute values, and applying the triangle inequality we obtain

$$\begin{aligned} \left|\delta_\eta(x)\right| &\leqslant \frac{1}{2\lambda}\int_{-\infty}^{x} \exp\left(-2\eta\left|x-y\right|\right)\ \left|\sigma(y)\right|\ dy \\ &+ \frac{1}{4\lambda}\left|\int_{-\infty}^{\infty} \exp(-2\eta\left|x-y\right|)\exp(2\lambda i y)\ \sigma(y)\ dy\right| \end{aligned} \tag{159}$$

for all $x \in \mathbb{R}$. By inserting (158) into (157) and taking absolute values, we conclude that

$$\begin{aligned} \left|\delta_\eta'(x)\right| &\leqslant \int_{-\infty}^{x} \exp\left(-2\eta\left|x-y\right|\right)\ \left|\sigma(y)\right|\ dy \\ &+ \frac{1}{2}\left|\int_{-\infty}^{\infty} \exp(-2\eta\left|x-y\right|)\exp(2\lambda i y)\ \sigma(y)\ dy\right| \end{aligned} \tag{160}$$

for all $x \in \mathbb{R}$. We now combine Lemma 3 with our assumption that

$$\lambda \geqslant \frac{2}{\mu}\log\left(\frac{24\Gamma}{\eta}\right). \tag{161}$$

which is part of (151), to see that

$$\left|\int_{-\infty}^{\infty} \exp(-2\eta\left|x-y\right|)\exp(2\lambda i y)\ \sigma(y)\ dy\right| \leqslant \frac{3\Gamma}{2}\exp\left(-\frac{\mu\lambda}{2}\right) \leqslant \frac{\eta}{16} \tag{162}$$

for all $x \in \mathbb{R}$. We insert (162) into (159) and (160) in order to obtain the inequalities

$$\left|\delta_\eta(x)\right| \leqslant \frac{1}{2\lambda}\int_{-\infty}^{x} \exp\left(-2\eta\left|x-y\right|\right)\ \left|\sigma(y)\right|\ dy + \frac{\eta}{64\lambda} \quad \text{for all} \quad x \in \mathbb{R} \tag{163}$$

and

$$\left|\delta_\eta'(x)\right| \leqslant \int_{-\infty}^{x} \exp\left(-2\eta\left|x-y\right|\right)\ \left|\sigma(y)\right|\ dy + \frac{\eta}{32} \quad \text{for all} \quad x \in \mathbb{R}. \tag{164}$$

Now we define δ via the formula

$$\delta(x) = T\left[\sigma\right](x). \tag{165}$$

From (78) and (165) we conclude that

$$\sigma(x) = S\left[\delta\right](x) + p(x) + \nu(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{166}$$

We combine conclusion (81) of Theorem 3 with the assumption that

$$\lambda \geqslant \frac{1}{\mu}\log\left(\frac{16\Gamma}{\mu\eta^2}\right) \tag{167}$$

which is part of (151), in order to conclude that

$$\|v\|_\infty \leqslant \frac{\Gamma}{\mu}\exp(-\mu\lambda) \leqslant \frac{\eta^2}{16}. \tag{168}$$

We combine (152), (168) and the fact that

$$\int_{-\infty}^{x} \exp(-2\eta\,|x-y|)dy = \frac{1}{2\eta} \quad \text{for all} \quad x \in \mathbb{R} \tag{169}$$

in order to conclude that

$$\int_{-\infty}^{x} \exp(-2\eta\,|x-y|)\left(|p(y)| + |\nu(y)|\right)\, dy \leqslant \frac{\eta}{16} \tag{170}$$

for all $x \leqslant a$. We combine Lemma 2 with (169) in order to conclude that

$$\int_{-\infty}^{x} \exp\left(-2\eta\,|x-y|\right)|S\left[\delta\right](y)|\, dy \leqslant \frac{\eta}{136} + \frac{102\lambda^2}{50\eta}\sup_{-\infty\leqslant y\leqslant x}|\delta_\eta(y)|^2 + \frac{1}{4\eta}\sup_{-\infty\leqslant y\leqslant x}\left|\delta_\eta'(y)\right|^2 \tag{171}$$

for all $x \in \mathbb{R}$. By combining (166), (170) and (171) we conclude that

$$\int_{-\infty}^{x} \exp(-2\eta\,|x-y|)\,|\sigma(y)|\, dy \leqslant \frac{19\eta}{272} + \frac{102\lambda^2}{50\eta}\sup_{-\infty\leqslant y\leqslant x}|\delta_\eta(x)|^2 + \frac{1}{4\eta}\sup_{-\infty\leqslant y\leqslant x}\left|\delta_\eta'(x)\right|^2 \tag{172}$$

for all $x \leqslant a$. Inserting (172) into (163) and (164) yields the inequalities

$$|\delta_\eta(x)| \leqslant \frac{55\eta}{1088\lambda} + \frac{102\lambda}{100\eta}\sup_{-\infty\leqslant y\leqslant x}|\delta_\eta(x)|^2 + \frac{19}{272\eta\lambda}\sup_{-\infty\leqslant y\leqslant x}\left|\delta_\eta'(x)\right|^2 \tag{173}$$

and

$$\left|\delta_\eta'(x)\right| \leqslant \frac{9\eta}{136} + \frac{102\lambda^2}{50\eta}\sup_{-\infty\leqslant y\leqslant x}|\delta_\eta(x)|^2 + \frac{1}{4\eta}\sup_{-\infty\leqslant y\leqslant x}\left|\delta_\eta'(x)\right|^2, \tag{174}$$

both of which hold for all $x \leqslant a$.

We denote by Ω the set

$$\left\{x \leqslant a : |\delta_\eta(y)| \leqslant \frac{\eta}{4\lambda} \quad \text{and} \quad \left|\delta_\eta'(y)\right| \leqslant \frac{\eta}{4} \quad \text{for all} \quad y \in (-\infty, x]\right\}. \tag{175}$$

The continuity of δ_η and δ_η' imply that Ω is closed in the relative topology of $(-\infty, a]$ (that is, the topology it inherits as a subset of $\mathbb{R}$). Conclusion (110) of Lemma 1 implies that Ω is nonempty. We let $x^* < a$ denote an element of Ω. By inserting the inequalities

$$|\delta_\eta(y)| \leqslant \frac{\eta}{4\lambda} \quad \text{for all} \quad y \leqslant x^* \tag{176}$$

and

$$\left|\delta_\eta'(y)\right| \leqslant \frac{\eta}{4} \quad \text{for all} \quad y \leqslant x^* \tag{177}$$

into (173) and (174) we conclude that

$$|\delta_\eta(x)| \leqslant \frac{55\eta}{1088\lambda} + \frac{102\eta}{1600\lambda} + \frac{\eta}{128\lambda} = \frac{6643\eta}{54400\lambda} < \frac{\eta}{4\lambda} \tag{178}$$

and

$$\left|\delta_\eta'(x)\right| \leqslant \frac{19\eta}{272} + \frac{102\eta}{800} + \frac{\eta}{64} = \frac{5693\eta}{27200} < \frac{\eta}{4} \tag{179}$$

for all $x \leqslant x^*$. The continuity of δ_η and δ_η' together with (178) and (179) imply an open neighborhood of x^* is contained in Ω. In other words, Ω is open in the relative topology of $(-\infty, a]$. In fact, Ω must be all of $(-\infty, a]$ since it is a nonempty set which is both open and closed in the relative topology of the connected set $(-\infty, a]$. The conclusions (154) and (155) follow immediately. □

We now combine Lemmas 1, 2, 3 and 4 in order to establish the principal result of this section, which is a bound on the restriction of the nonoscillatory solution δ of the nonlinear differential equation

$$\delta''(x) + 4\lambda^2\delta(x) = S[\delta](x) + p(x) + \nu(x) \tag{180}$$

to the interval $(-\infty, a]$ under the assumption that p is of small magnitude there.

Theorem 4. *Suppose that the hypotheses of Theorem 3 are satisfied, and that σ and ν are the functions obtained by invoking it. Suppose further that C_1 is the real number*

$$C_1 = \max\left\{24\Gamma, \sqrt{\frac{16\Gamma}{\mu}}\right\}, \tag{181}$$

and that

$$\lambda \geqslant \max\left\{4\Gamma, \frac{4}{\mu}, \frac{\Gamma}{4\mu}, \frac{2}{\mu}W_0\left(C_1\mu\right)\right\}. \tag{182}$$

Suppose also that a is a real number, that

$$|p(x)| \leqslant \frac{C_1^2}{16}\exp\left(-\mu\lambda\right) \quad \text{for all} \quad x \leqslant a, \tag{183}$$

and that δ is defined via the formula

$$\delta(x) = T[\sigma](x). \tag{184}$$

Then

$$|\delta(x)| \leqslant \frac{C_1}{2\lambda}\exp\left(-\frac{\mu\lambda}{2}\right) \tag{185}$$

and

$$\left|\delta'(x)\right| \leqslant \frac{C_1}{2}\exp\left(-\frac{\mu\lambda}{2}\right) \tag{186}$$

for all $x \leqslant a$.

Remark 2. *In (182), W_0 refers to the branch of the Lambert W function which is greater than or equal to -1 on the interval $[-1/e, \infty)$; see Section 2.5.*

Proof. We let

$$\eta = C_1 \exp\left(-\frac{\mu\lambda}{2}\right). \tag{187}$$

From our assumption that

$$\lambda \geqslant \frac{2}{\mu}W_0\left(C_1\mu\right), \tag{188}$$

which is part of (182), and the inequality (46) of Section 2.5 we conclude that

$$\lambda \geqslant 2C_1\exp\left(\frac{-\mu\lambda}{2}\right) = 2\eta. \tag{189}$$

Moreover, by inserting (181) into (189) and using the inequality (46) we obtain

$$\eta \leqslant 24\Gamma\exp\left(\frac{-\mu\lambda}{2}\right). \tag{190}$$

and

$$\eta^2 \leqslant \frac{16\Gamma}{\eta}\exp\left(-\mu\lambda\right). \tag{191}$$

It follows immediately from (190) and (191) that

$$\lambda \geqslant \max\left\{\frac{2}{\mu}\log\left(\frac{24\Gamma}{\eta}\right), \frac{1}{\mu}\log\left(\frac{16\Gamma}{\mu\eta^2}\right)\right\}. \tag{192}$$

Together (187), (192) and (182) ensure that the hypothesis (151) of Lemma 4 is satisfied. From (187) and (183), we conclude that

$$|p(x)| \leqslant \frac{\eta^2}{16}, \tag{193}$$

so the hypothesis (152) of Lemma 4 is satisfied as well. By invoking Lemma 4 we see that

$$|\delta_\eta(x)| \leqslant \frac{\eta}{4\lambda} \tag{194}$$

and

$$\left|\delta_\eta'(x)\right| \leqslant \frac{\eta}{4} \tag{195}$$

for all $x \leqslant a$. Inserting (187) into (194) and (195) gives the inequalities

$$|\delta_\eta(x)| \leqslant \frac{C_1}{4\lambda}\exp\left(-\frac{\mu\lambda}{2}\right) \tag{196}$$

and

$$\left|\delta_\eta'(x)\right| \leqslant \frac{C_1}{4}\exp\left(-\frac{\mu\lambda}{2}\right), \tag{197}$$

which hold for all $x \leqslant a$. We combine the hypotheses (76) of Theorem 3, and conclusions (113) and (114) of Lemma 1 to obtain

$$|\delta(x) - \delta_\eta(x)| \leqslant \frac{3\Gamma\eta}{\pi\mu\lambda^2} \leqslant \frac{3}{16\pi}C_1\exp\left(-\frac{\mu\lambda}{2}\right). \tag{198}$$

Similarly, from (182) and conclusions (113) and (114) of Lemma 1 we obtain

$$\left|\delta'(x) - \delta_\eta'(x)\right| \leqslant \frac{3\Gamma\eta}{\mu^2\pi\lambda^2} \leqslant \frac{3}{16\pi}C_1\exp\left(-\frac{\mu\lambda}{2}\right). \tag{199}$$

We combine (198) with (196) to obtain (185), and (199) with (197) to obtain (186). □

3.4. A continuity result

Theorem 4 implies that when the magnitude of the function p is sufficently small on the interval $(-\infty, a]$, the values of $\delta(a)$ and $\delta'(a)$ are on the order of

$$\exp\left(-\frac{1}{2}\mu\lambda\right). \tag{200}$$

Moreover, the magnitude of the function ν appearing in (86) is on the order of

$$\exp\left(-\mu\lambda\right). \tag{201}$$

The following theorem follows from these observations and standard results regarding the continuity of ordinary differential equations with respect to the perturbation of initial values and coefficients (see, for instance, [6] or [27]).

Theorem 5. *Suppose that the hypotheses of Theorem 3 are satisfied, that σ and ν are the functions obtained by invoking it, and that δ is the function defined via the formula*

$$\delta(x) = T[\sigma](x) \tag{202}$$

so that δ *solves the equation*

$$\delta''(x) + 4\lambda^2\delta(x) = S[\delta](x) + p(x) + \nu(x) \quad \textit{for all} \;\; x \in \mathbb{R}. \tag{203}$$

Suppose also that $\eta : [a, b] \to \mathbb{R}$ *is a continuously differentiable function, and that there exists a constant* $C > 0$ *such that*

$$|\eta(x)| + |\eta'(x)| \leqslant C \exp\left(-\frac{\mu\lambda}{2}\right) \tag{204}$$

for all $a \leqslant x \leqslant b$*. Then there exist a constant* $C' > 0$ *and a twice continuously differentiable function* $\delta_0 : [a, b] \to \mathbb{R}$ *such that* δ_0 *solves the initial value problem*

$$\begin{cases} \delta_0''(x) + 4\lambda^2\delta_0(x) = S[\delta_0](x) + p(x) + \eta(x) \quad \textit{for all} \;\; x_0 \leqslant x \leqslant x_1 \\ \delta_0(a) = 0 \\ \delta_0'(b) = 0, \end{cases} \tag{205}$$

and

$$|\delta(x) - \delta_0(x)| + |\delta'(x) - \delta_0'(x)| \leqslant C' \exp\left(-\frac{\mu\lambda}{2}\right) \tag{206}$$

for all $a \leqslant x \leqslant b$*.*

That the difference between the solution r_2 of the boundary value problem (19) and the nonoscillatory solution r of (16) is on the order of $\exp\left(-\frac{\mu\lambda}{2}\right)$ follows easily from Theorem 5.

4. Numerical algorithm

In this section, we describe an algorithm for the solution of the boundary value problem

$$\begin{cases} y''(t) + \lambda^2 q(t)y(t) = 0 \;\; \text{for all} \;\; a \leqslant t \leqslant b \\ c_1 y(a) + c_2 y'(a) = \alpha \\ c_3 y(b) + c_4 y'(b) = \beta. \end{cases} \tag{207}$$

where c_1, c_2, c_3, c_4, α, β and $\lambda > 0$ are real numbers, and q is strictly positive on the interval $[a, b]$ and analytic in an open set containing the interval $[a, b]$. It can be easily modified to address, *inter alia*, initial value problems.

The algorithm exploits the analytical appparatus developed in Section 3 in order to construct a solution r_2 of the logarithm form of Kummer's equation

$$r_2''(t) - \frac{1}{4}(r_2'(t))^2 + 4\lambda^2\left(\exp(r_2(t)) - q(t)\right) = 0 \quad \text{for all} \quad a \leqslant t \leqslant b. \tag{208}$$

Once the function r_2 has been obtained, we construct a phase function α via the formula

$$\alpha(t) = \gamma \int_0^t \exp\left(\frac{r_2(u)}{2}\right) \, du. \tag{209}$$

It has the property that the functions u, v defined by the formulas

$$u(t) = \frac{\cos(\alpha(t))}{|\alpha'(t)|^{1/2}} \tag{210}$$

and

$$v(t) = \frac{\sin(\alpha(t))}{|\alpha'(t)|^{1/2}} \tag{211}$$

form a basis in the space of solutions of the ordinary differential equation

$$y''(t) + \lambda^2 q(t)y(t) = 0 \;\; \text{for all} \;\; a \leqslant t \leqslant b. \tag{212}$$

We compute real numbers d_1 and d_2 such that the function

$$y(t) = d_1 u(t) + d_2 v(t) \tag{213}$$

satisfies the boundary conditions

$$\begin{aligned} c_1 y(a) + c_2 y'(a) &= \alpha \\ c_3 y(b) + c_4 y'(b) &= \beta \end{aligned} \tag{214}$$

by solving the system of two linear algebraic equations in the two unknowns d_1, d_2 obtained by inserting (213) into (214).

In addition to the value of λ and a routine for evaluating the function q at any point on the interval $[a, b]$, the user supplies as inputs to the algorithm an integer $m > 0$ and a partition

$$a = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_n = b \tag{215}$$

of the interval $[a, b]$. For each $j = 1, \ldots, n$, the restrictions of the functions r and α to $[\xi_{j-1}, \xi_j]$ are represented by their values at the points

$$x_{j,0}, x_{j,1}, x_{j,2}, \ldots, x_{j,m}, x_{j,m} \tag{216}$$

of the $(m + 1)$-point Chebyshev grid on the interval $[\xi_{j-1}, \xi_j]$ (see Section 2.6). The assumption is, of course, that the restrictions of these functions to each subinterval are well-approximated by polynomials of degree m. Note that for each $j = 1, \ldots, n - 1$, the last Chebyshev point in the interval $[\xi_{j-1}, \xi_j]$ coincides with the first Chebyshev point in the interval $[\xi_j, \xi_{j+1}]$; that is,

$$x_{j,m} = \xi_{j+1} = x_{j+1,0} \tag{217}$$

for all $j = 1, \ldots, n - 1$.

The output of the algorithm consists of the values of α and α' at each of the the $nm + 1$ points

$$x_{1,0}, \ldots, x_{1,m}, x_{2,0}, \ldots, x_{2,m}, \ldots x_{n,0}, \ldots, x_{n,m} \tag{218}$$

(there are $nm + 1$ points rather than $n(m + 1)$ points because $x_{1,m} = x_{2,0}$, $x_{2,m} = x_{3,0}$, etc.). Using this data, the value of the solution y_0 of the boundary value problem (207) can be computed at any point t in $[a, b]$. More specifically, to evaluate $y_0(t)$ at the point t, we calculate $\alpha(t)$ and $\alpha'(t)$ via Chebyshev interpolation (as discussed in Section 2.6), then evaluate $u(t)$ and $v(t)$ using formulas (210) and (211), and finally insert the values of $u(t)$ and $v(t)$ into (213) in order to obtain $y_0(t)$.

Our algorithm calls for solving a number of stiff ordinary differential equations. In our implementation, we used the spectral deferred correction method described in [10]. It was chosen for its excellent stability properties; however, any standard approach to the numerical solution of stiff ordinary differential equation can be substituted for the algorithm of [10].

We now describe the procedure for the construction of the phase function α in detail. It consists of the following four phases.

Phase 1: Construction of the Windowed Problem

In the first phase of the algorithm we construct a windowed version $\tilde{q}$ of the function q using the following sequence of steps:

1. We let

$$\psi(t) = \frac{1 - \operatorname{erf}\left(\frac{13}{b-a}\left(t - \frac{a+b}{2}\right)\right)}{2} \tag{219}$$

so that $\psi(t) \approx 1$ for all t near a and $\psi(t) \approx 0$ for all t near b. Note that the constant 13 in (219) was chosen to be the smallest positive integer such that the quantities $|1 - \phi(a)|$ and $|\phi(b)|$ are less than machine precision.

2. We define the function $\tilde{q}$ by the formula

$$\tilde{q}(t) = \psi(t) + (1 - \psi(t))q(t) \tag{220}$$

so that $\tilde{q}(t) \approx 1$ when t is close to a and $\tilde{q}(t) \approx q(t)$ when t is close to b. We refer to $\tilde{q}$ as the windowed version of q.

Phase 2: Solution of the windowed problem

In this phase, we solve the initial value problem

$$\begin{cases} r_1''(t) - \dfrac{1}{4}\left(r_1'(t)\right)^2 + 4\lambda^2\left(\exp(r_1(t)) - \tilde{q}(t)\right) = 0 \quad \text{for all} \quad a \leqslant t \leqslant b \\ r_1(a) = r_1'(a) = 0, \end{cases} \tag{221}$$

with the windowed function $\tilde{q}$ is in place of the original function q. We denote by $\tilde{r}$ the nonoscillatory solution of the logarithm form of Kummer's equation obtained by applying Theorem 3 to the second order ordinary differential equation

$$y''(t) + \lambda^2\tilde{q}(t)y(t) = 0. \tag{222}$$

According to the discussion in Section 3,

$$|r_1(t) - \tilde{r}(t)| + \left|r_1'(t) - \tilde{r}'(t)\right| = \mathcal{O}\left(\exp\left(-\frac{\lambda\mu}{2}\right)\right) \tag{223}$$

for all t close to b. Assuming that λ is sufficiently large, the difference between r_1 and the nonoscillatory function $\tilde{r}$ is well below machine precision and r_1 can be treated as nonoscillatory for the purposes of numerical computation.

For each $j = 1, \ldots, n$, we compute the solution of (221) at the points

$$x_{j,0}, \ldots, x_{j,m} \tag{224}$$

of the $(m+1)$-point Chebyshev grid on $[\xi_{j-1}, \xi_j]$. If $j = 1$, then the initial conditions are taken to be

$$r_1(a) = r_1'(a) = 0. \tag{225}$$

If, on the other hand, $j > 1$, then we enforce the conditions

$$r_1\left(x_{j,1}\right) = r\left(x_{j-1,m}\right) \tag{226}$$

and

$$r_1'\left(x_{j,1}\right) = r_1'\left(x_{j-1,m}\right); \tag{227}$$

that is, we require that r_1 and its first derivative agree at the left endpoint of the interval with the value and derivative of the solution at the right endpoint of the previous interval.

Phase 3: Solution of the original problem

In this phase, we solve the problem

$$\begin{cases} r_2''(t) - \dfrac{1}{4}\left(r_2'(t)\right)^2 + 4\lambda^2\left(\exp(r_2(t)) - q(t)\right) = 0 \quad \text{for all} \quad a \leqslant t \leqslant b \\ r_2(b) = r_1(b) \\ r_2'(b) = r_1'(b) \end{cases} \tag{228}$$

The intervals are processed in decreasing order: the n^{th} interval $[\xi_{n-1}, \xi_n]$ is the first to be processed, then $[\xi_{n-2}, \xi_{n-1}]$, and so on. Boundary conditions are imposed at the left end point of each interval;

in particular, when processing the n^{th} interval we require that

$$\begin{aligned} r_2(\xi_n) &= r_1(\xi_n) \\ r_2'(\xi_n) &= r_1(\xi_n) \end{aligned} \tag{229}$$

and while processing each of the subsequent intervals $[\xi_{j-1}, \xi_j]$ we require that

$$\begin{aligned} r_2(x_{j,m}) &= r_2(x_{j+1,0}) \\ r_2'(x_{j,m}) &= r_2'(x_{j+1,0}). \end{aligned} \tag{230}$$

According to the discussion in Section 3,

$$|r(t) - r_2(t)| = \mathcal{O}\left(\exp\left(-\frac{\mu\lambda}{2}\right)\right) \tag{231}$$

for all $a \leqslant t \leqslant b$. As in the case of r_1, (231) implies that in the high-frequency regime, the difference between r_2 and the nonoscillatory solution r of the logarithm form of Kummer's equation associated with the coefficient q is much smaller than machine precision. Consequently, we regard r_2 as nonoscillatory for the purposes of numerical computation.

Phase 4: Preparation of the output

In this final phase, the values of the functions α and α' are tabulated at each of the points (218) via the following sequence of steps:

1. We compute the values of α' at the points (218) using the formula

$$\alpha'(t) = \lambda \exp\left(\frac{r_2'(t)}{2}\right). \tag{232}$$

2. For each $j = 1, \ldots, n$, we apply the spectral integration matrix of order m (see Section 2.6) to the vector

$$\begin{pmatrix} \alpha'(x_{j,0}) \\ \alpha'(x_{j,1}) \\ \vdots \\ \alpha'(x_{j,m}) \end{pmatrix} \tag{233}$$

in order to obtain the values

$$\alpha_j(x_{j,0}), \alpha_j(x_{j,1}), \ldots, \alpha_j(x_{j,m}) \tag{234}$$

of an antiderivative α_j of the restriction of α' to the interval $[\xi_{j-1}, \xi_j]$ at the nodes of the $(m+1)$-point Chebyshev grid on that interval. Note that the value of $\alpha_j(\xi_j)$ is not necessarily consistent with the value of $\alpha_{j+1}(\xi_j)$. This problem is corrected in the following steps.

3. For each $j = 1, \ldots, n$, we define a real number γ_j as follows

$$\begin{cases} \gamma_j = \alpha(\xi_{1,0}) & \text{if } j = 1 \\ \gamma_j = \alpha(\xi_{j-1,m}) & \text{if } j > 1 \end{cases} \tag{235}$$

4. For each $j = 2, \ldots, n$ and each $i = 0, \ldots, m$, the value of the phase function α at the point $x_{j,i}$ is computed via the formula

$$\alpha(x_{j,i}) = \alpha_j(x_{j,i}) - \alpha_j(x_{j,0}) + \gamma_j. \tag{236}$$

The output of the algorithm consists of the values of α' at the nodes (216) computed in Step 1 of Phase 4 and the values of α at the nodes (216) computed in Step 4 of Phase 4.

5. Numerical experiments

In this section, we describe numerical experiments performed to evaluate the performance of the algorithm of Section 4. Our code was written in Fortran and compiled with the Intel Fortran Compiler version 13.1.3. All calculations were carried out on a desktop computer equipped with an Intel Xeon X5690 CPU running at 3.47 GHz. Unless otherwise noted, double precision (Fortran REAL*8) arithmetic was used.

5.1. Comparison with a standard solver

We measured the performance of the algorithm of this paper by applying it to the initial value problem

$$\begin{cases} y''(t) + \lambda^2 q(t) y(t) = 0 \quad \text{for all} \quad -1 \leqslant t \leqslant 1 \\ y(-1) = 0 \\ y'(-1) = \lambda, \end{cases} \tag{237}$$

where q is defined by the formula

$$q(t) = 1 - t^2 \cos(3t), \tag{238}$$

for seven values of λ. A reference solution was obtained by executing the spectral deferred correction method of [10] in extended precision (Fortran REAL*16) arithmetic. The interval $[-1, 1]$ was partitioned into 10 equispaced subintervals and the 16 point Chebyshev grid was used to represent the nonoscillatory phase function on each subinterval. For each value of λ, the obtained solution was compared to the reference solution at 1000 randomly chosen points on the interval $[-1, 1]$.

The results of this experiment are reported in Table 1. Each row there corresponds to one value of λ and reports the time required to construct the nonoscillatory phase function, the average time required to evaluate the solution of (237) using this nonoscillatory phase function, and the maximum absolute error which was observed. We see that the time required to solve (237) was independent of the value of the parameter λ, and that the obtained accuracy decreased as λ increased. This loss of precision was incurred when the sine and cosine of large arguments were calculated in the course of evaluating the functions u, v defined via formulas (2), (3).

Plots of the function q defined by (238) and the windowed version of q constructed as an intermediate step by the algorithm of Section 4 are shown in Figure 1. Plots of the solution r of the logarithm form of Kummer's equation when $\lambda = 10^7$ and the windowed version r_1 of r constructed as an intermediate step by the algorithm of Section 4 are shown in Figure 2.

5.2. Phase functions for Chebyshev's equation

Chebyshev's equation

$$(1 - t^2) y''(t) - t y'(t) + \lambda^2 y(t) = 0 \quad \text{for all} \quad -1 \leqslant t \leqslant 1 \tag{239}$$

admits an exact nonoscillatory phase function which can be represented via elementary functions. More specifically,

$$\alpha_0(t) = \lambda \arccos(t) \tag{240}$$

is a nonoscillatory phase function for the second order equation

$$\psi''(t) + \left(\frac{2 + t^2 + 4\lambda^2 \left(1 - t^2\right)}{4\left(1 - t^2\right)^2} \right) \psi(t) = 0 \quad \text{for all} \quad -1 \leqslant t \leqslant 1 \tag{241}$$

obtained by introducing

$$\psi(t) = (1 - t^2)^{1/4} y(t) \tag{242}$$

into (239). For each $\lambda = 10, 20, \ldots, 1000$, we applied the algorithm of Section 4 to (241) and compared the resulting phase function to (240). Figure 3 displays a plot of the relative difference

$$\frac{\|\alpha - \alpha_0\|_\infty}{\|\alpha_0\|_\infty} \tag{243}$$

between the exact phase function α_0 and the phase function α obtained via the algorithm of Section 4 as a function of λ. We observe that as λ increases, the difference between the phase function obtained via the algorithm and the function $\lambda \arccos(t)$ decays at an exponential rate.

5.3. Evaluation of Bessel functions.

We compared the cost of evaluating Bessel functions of integer order via the standard recurrence relation with that of doing so using a nonoscillatory phase function.

We denote by J_ν the Bessel function of the first kind of order ν. It is a solution of the second order differential equation

$$t^2 y''(t) + t y'(t) + (t^2 - \nu^2) y(t) = 0, \tag{244}$$

which is brought into the standard form

$$\psi''(t) + \left(1 - \frac{\lambda^2 - 1/4}{t^2}\right) \psi(t) = 0 \tag{245}$$

via the transformation

$$\psi(t) = \sqrt{t}\, y(t). \tag{246}$$

An inspection of (245) reveals that J_ν is nonoscillatory on the interval

$$\left(0, \frac{1}{2}\sqrt{4\nu^2 - 1}\right) \tag{247}$$

and oscillatory on the interval

$$\left(\frac{1}{2}\sqrt{4\nu^2 - 1}, \infty\right). \tag{248}$$

In addition to being a solution of a second order differential equation, the Bessel function of the first kind of order ν satisfies the three-term recurrence relation

$$J_{\nu+1}(t) = \frac{2\nu}{t} J_\nu(t) - J_{\nu-1}(t). \tag{249}$$

The recurrence (249) is numerically unstable in the forward direction; however, when evaluated in the direction of decreasing index, it yields a stable mechanism for evaluating Bessel functions of integer order (see, for instance, Chapter 3 of [21]). These and many other properties of Bessel functions are discussed in [26].

For each of 8 values of n, we obtained an approximation of the Bessel function J_n via the algorithm of Section 4 and compared its values to those obtained through the recurrence relation at a collection of 1000 randomly chosen points in the interval $[\frac{1}{2}\sqrt{4n^2 - 1}, 10n]$. The results of this experiment are shown in Table 2. The phase function produced by the algorithm of Section 4 when $n = 10^4$ is shown in Figure 4.

5.4. Evaluation of Legendre functions

We used the algorithm of this paper to evaluate Legendre functions of the first kind of various orders on the interval $[-1, 1]$.

For each real number ν, we denote by P_ν the Legendre function of the first kind of order ν. It is the solution of Legendre's equation

$$(1-t^2)y''(t) - 2ty'(t) + \nu(\nu+1)y(t) = 0 \quad \text{for all} \quad -1 \leqslant t \leqslant 1 \tag{250}$$

which is regular at the origin. Letting

$$\psi(t) = \sqrt{1-t^2}\, y(t) \tag{251}$$

in Equation (250) yields

$$\psi''(t) + \left(\frac{1}{(1-t^2)^2} + \frac{\nu}{1-t^2} + \frac{\nu^2}{(1-t^2)^2} \right) \psi(t) = 0 \quad \text{for all} \quad -1 \leqslant t \leqslant 1, \tag{252}$$

which is in a suitable form for the algorithm of Section 4.

We observe that the coefficient in equation (252) is singular at ± 1, which means that phase functions for Legendre's equation are singular at ± 1 as well. Accordingly, in this experiment we used as input to the algorithm of Section 4 a partition of the form

$$-\xi_{-k} < -\xi_{-k+1} < \ldots < \xi_{-1} < \xi_0 < \xi_1 < \ldots \xi_{k-1} < \xi_k, \tag{253}$$

where $k = 50$ and ξ_j is defined by the formula

$$\xi_j = 1 - 2^{-|j|}. \tag{254}$$

The set $\{\xi_j\}$ is a "graded mesh" whose points cluster near the singularities ± 1 of the coefficient in (252). Note that (253) is not a partition of the entire interval $[-1, 1]$ but rather a partition of $[-b, b]$, where

$$b = 1 - 2^{-50}. \tag{255}$$

Functions were represented using 16^{th} order Chebyshev expansions on each of the intervals $[\xi_j, \xi_{j+1}]$.

For each of 11 values of ν, the algorithm of this paper was applied to Equation (252) and the solution evaluated at a collection of 1000 randomly chosen points on the interval $[-1, 1]$. In order to assess the the error in each obtained solution, we constructed a reference solutions by performing the calculations a second time using extended precision (Fortran REAL*16) arithmetic. The results are reported in Table 3. Each row corresponds to one value of ν and reports the time required to construct the nonoscillatory phase functions, the average time required to evaluate the Legendre function of the first kind of order ν using this nonoscillatory phase function, and the maximum observed absolute error. Figure 5 depicts the solution of the logarithm form of Kummer's equation obtained by the algorithm of this paper when $\nu = \pi \cdot 10^5$.

5.5. Evaluation of prolate spheroidal wave functions

We used the algorithm of Section 4 to evaluate prolate spheroidal wave functions of order 0 and we compared its performance with that of the Osipov-Rokhlin algorithm [22].

Suppose that $c > 0$ is a real number. Then there exists a sequence

$$0 < \chi_{c,0} < \chi_{c,1} < \chi_{c,2} < \cdots \tag{256}$$

of positive real numbers such for each nonnegative integer n, the second order differential equation

$$(1-t^2)\psi''(t) - 2t\psi'(t) + (\chi_{c,n} - c^2t^2)\psi(t) = 0 \tag{257}$$

has a continuous solution on the interval $[-1, 1]$. These solutions are the prolate spheroidial wave functions of order 0 associated with the parameter c. We denote them by

$$\psi_{c,0}(t), \psi_{c,1}(t), \psi_{c,2}(t), \ldots. \tag{258}$$

The monograph [23] contains a detailed discussion of the prolate spheroidal wave functions of order 0.

By introducing the function

$$\varphi(t) = \psi(t)\sqrt{1-t^2} \tag{259}$$

into (257), we bring it into the form

$$\varphi''(t) + \left(\frac{1}{(1-t^2)^2} + \frac{\chi_{c,n}}{1-t^2} - c^2t^2 \right) \varphi(t) = 0. \tag{260}$$

An inspection of (260) reveals that the coefficient in (260) is nonnegative on the interval $[-1, 1]$ when $\chi_n \geqslant c^2$.

For several values of c and $\chi_{n,c} > c^2$, we evaluated the prolate spheroidial wave function $\psi_{c,n}$ at a collection of 100 randomly chosen points in the interval $[-1, 1]$ by applying the algorithm of Section 4 to (260) and via the Osipov-Rokhlin algorithm. Table 4 presents the results and Figure 6 shows a plot of $\alpha(t) - ct$, where $c = 10^5$, $n = 63769$, $\chi_{c,n} = 1.00060408908491 \times 10^{10}$ and α is the nonoscillatory phase function for Equation (260) produced by the algorithm of Section 4.

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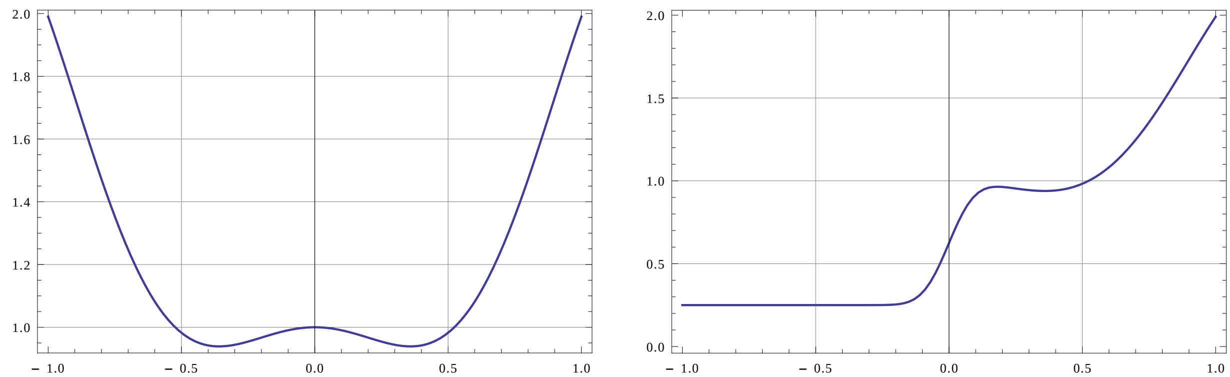

Figure 1: The function q defined by formula (238) in Section 5.1 (left) and the windowed version $\tilde{q}$ of q (right).

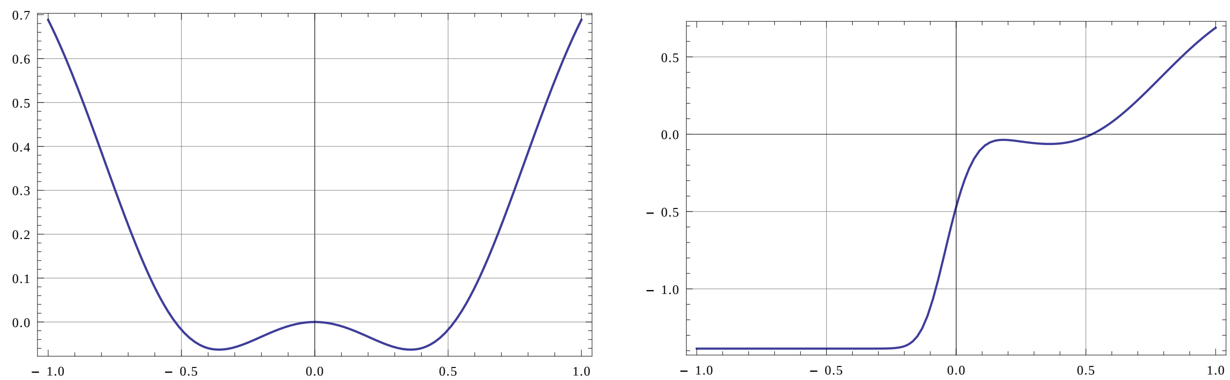

Figure 2: Plots of the solution r of the logarithm form of Kummer's equation associated with the ordinary differential equation (237) in Section 5.1 when $\lambda = 10^7$ (left) and the function r_1 constructed as an intermediate step by the algorithm of Section 4 (right).

λ	Phase function construction time	Avg. phase function evaluation time	Maximum error
10^1	7.25×10^{-02}	1.55×10^{-06}	6.93×10^{-14}
10^2	9.17×10^{-02}	1.58×10^{-06}	5.39×10^{-13}
10^3	6.74×10^{-02}	1.55×10^{-06}	3.01×10^{-12}
10^4	6.73×10^{-02}	1.55×10^{-06}	4.82×10^{-11}
10^5	6.72×10^{-02}	1.59×10^{-06}	3.23×10^{-10}
10^6	6.66×10^{-02}	1.64×10^{-06}	5.15×10^{-09}
10^7	8.60×10^{-02}	1.61×10^{-06}	3.64×10^{-08}

Table 1: The accuracy and running time of the algorithm of this paper when applied to the initial value problem (237) of Section 5.1.

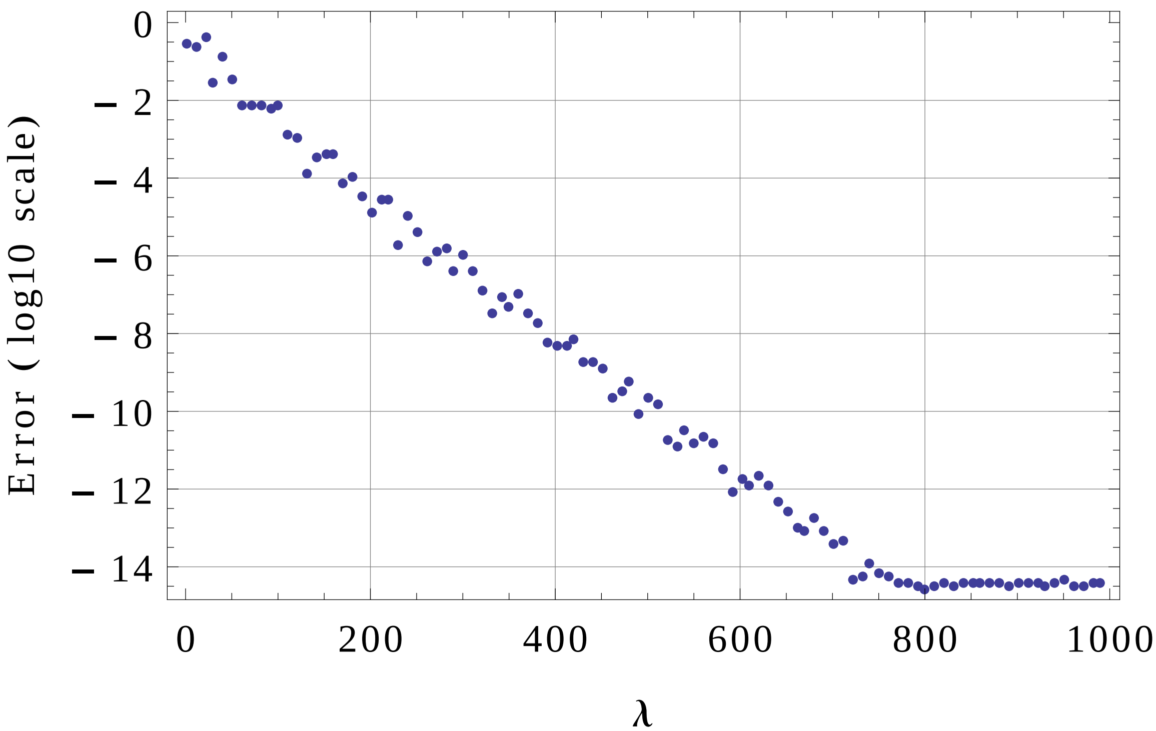

Figure 3: A plot of the base-10 logarithm of the relative difference between phase function obtained by applying the algorithm of this paper to Chebyshev's equation (241) and the well-known nonoscillatory phase function $\lambda \arccos(t)$ for Chebyshev's equation.

n	Phase function construction time	Avg. phase function evaluation time	Avg. recurrence evaluation time	Maximum error
10^1	1.70×10^{-02} secs	2.24×10^{-07} secs	1.40×10^{-06} secs	1.58×10^{-14}
10^2	2.27×10^{-02} secs	2.06×10^{-07} secs	6.17×10^{-06} secs	1.75×10^{-14}
10^3	1.62×10^{-02} secs	2.23×10^{-07} secs	4.60×10^{-05} secs	4.62×10^{-14}
10^4	1.65×10^{-02} secs	2.24×10^{-07} secs	4.29×10^{-04} secs	3.52×10^{-13}
10^5	1.62×10^{-02} secs	2.29×10^{-07} secs	4.12×10^{-03} secs	4.70×10^{-13}
10^6	1.66×10^{-02} secs	2.65×10^{-07} secs	4.20×10^{-02} secs	1.66×10^{-12}
10^7	2.94×10^{-02} secs	2.69×10^{-07} secs	4.22×10^{-01} secs	3.88×10^{-11}
10^8	6.42×10^{-01} secs	6.39×10^{-07} secs	$4.33\times10^{+00}$ secs	3.91×10^{-11}

Table 2: A comparison of the time required to evaluate the Bessel function J_n using the standard recurrence relation with that required to evaluate it using a nonoscillatory phase function. The recurrence relation approach scales as $O(n)$ in the order n while the time required by the phase function method is $O(1)$.

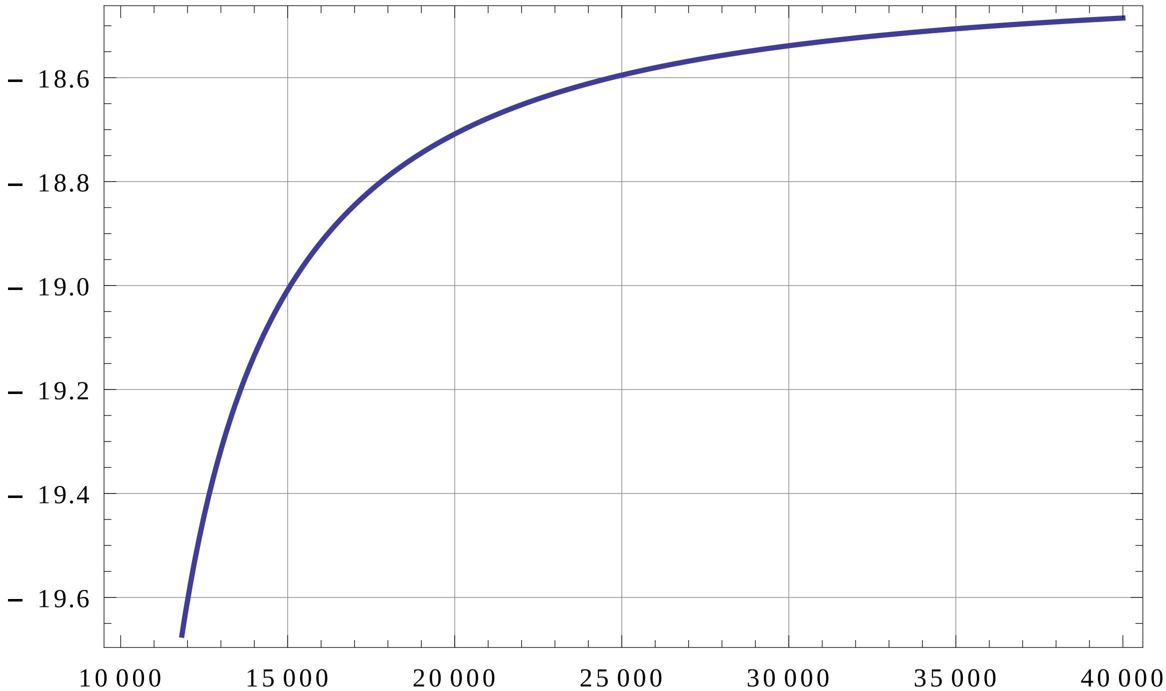

Figure 4: A plot of the nonoscillatory phase function for Bessel's equation (244) when $n = 10^4$.

ν	Phase function construction time	Avg. phase function evaluation time	Maximum error
$\pi \cdot 10^4$	2.92×10^{-03} secs	3.13×10^{-07} secs	1.32×10^{-13}
$\pi \cdot 10^5$	2.60×10^{-03} secs	3.84×10^{-07} secs	3.24×10^{-13}
$\pi \cdot 10^6$	2.46×10^{-03} secs	4.63×10^{-07} secs	1.09×10^{-12}
$\pi \cdot 10^7$	2.97×10^{-03} secs	3.67×10^{-07} secs	3.21×10^{-12}
$\pi \cdot 10^8$	2.53×10^{-03} secs	3.61×10^{-07} secs	1.35×10^{-11}
$\sqrt{2} \cdot 10^4$	5.61×10^{-03} secs	3.20×10^{-07} secs	7.22×10^{-13}
$\sqrt{2} \cdot 10^5$	5.43×10^{-03} secs	3.54×10^{-07} secs	3.32×10^{-12}
$\sqrt{2} \cdot 10^6$	2.49×10^{-03} secs	4.26×10^{-07} secs	1.13×10^{-12}
$\sqrt{2} \cdot 10^7$	2.61×10^{-03} secs	3.84×10^{-07} secs	2.79×10^{-12}
$\sqrt{2} \cdot 10^8$	5.43×10^{-03} secs	3.67×10^{-07} secs	8.65×10^{-12}
$\sqrt{2} \cdot 10^9$	4.94×10^{-03} secs	3.95×10^{-07} secs	2.85×10^{-11}

Table 3: The results obtained by applying the algorithm of Section 4 to Legendre's differential equation (250). We observe that the running time is independent of ν, but that some accuracy is lost when evaluating Legendre functions of large orders.

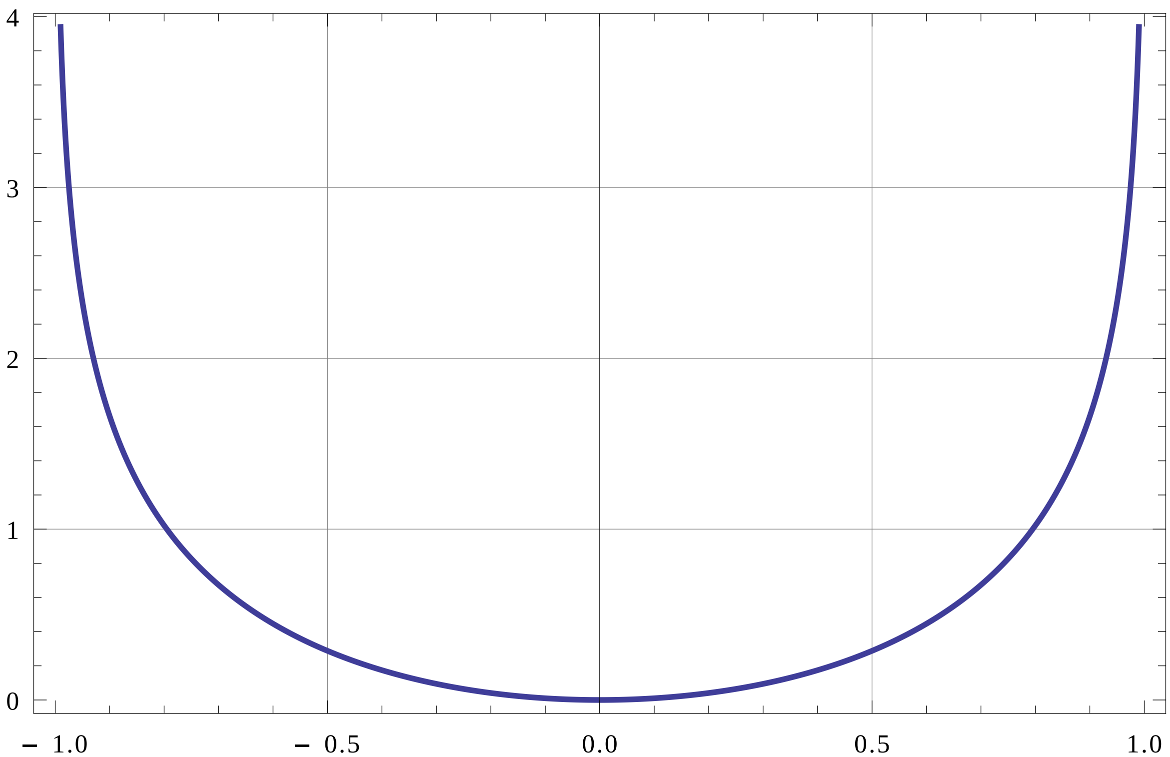

Figure 5: A plot of the nonoscillatory solution of the logarithm form of Kummer's equation associated with Legendre's differential equation (250) when $\nu = \pi \cdot 10^5$. This function has singularities at the points ± 1 and is represented using a graded mesh which becomes dense near them.

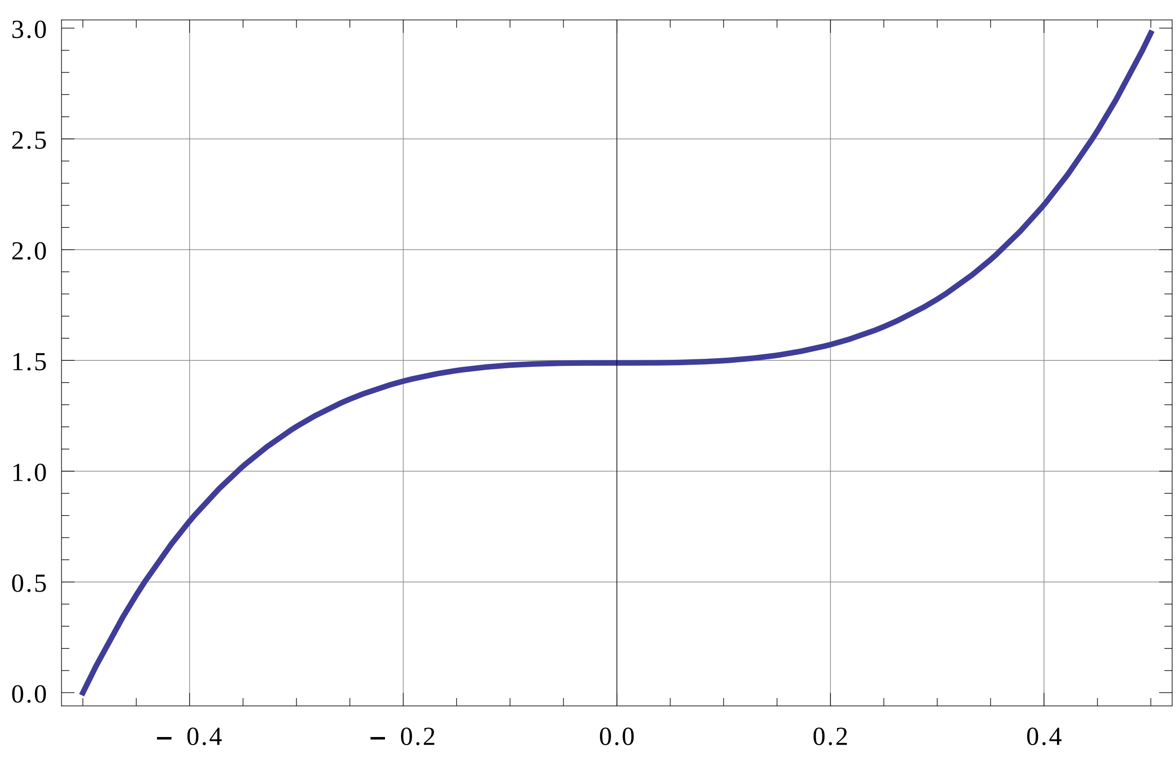

Figure 6: A plot of the function $\alpha(t) - ct$, where α is nonoscillatory phase function associated with equation (260), $c = 10^5$, $n = 63769$, and $\chi_{n,c} = 1.00060408908491 \times 10^{10}$.

c	n	$\chi_{c,n}$	Phase function construction time	Avg. phase function evaluation time	Average Osipov-Rokhlin evaluation time	Maximum error
10^4	12904	$2.18416195669669 \times 10^{+08}$	3.30×10^{-02} secs	3.22×10^{-07} secs	1.65×10^{-04} secs	3.24×10^{-11}
10^5	63769	$1.00060408908491 \times 10^{+10}$	3.29×10^{-02} secs	3.29×10^{-07} secs	1.11×10^{-03} secs	1.67×10^{-10}
10^5	95653	$1.44996988449419 \times 10^{+10}$	3.32×10^{-02} secs	3.19×10^{-07} secs	1.34×10^{-03} secs	2.00×10^{-10}
10^6	636748	$1.00005993679849 \times 10^{+12}$	3.31×10^{-02} secs	3.70×10^{-07} secs	1.14×10^{-02} secs	1.77×10^{-09}
10^6	1910244	$4.15761057502686 \times 10^{+12}$	3.30×10^{-02} secs	3.60×10^{-07} secs	2.34×10^{-02} secs	5.19×10^{-09}
10^7	12732696	$2.14063766093698 \times 10^{+14}$	3.08×10^{-02} secs	3.60×10^{-07} secs	1.59×10^{-01} secs	3.74×10^{-08}
10^8	127324798	$2.14057058422053 \times 10^{+16}$	4.34×10^{-02} secs	4.10×10^{-07} secs	$1.59\times10^{+00}$ secs	3.30×10^{-07}
10^8	190986447	$4.15616226165926 \times 10^{+16}$	4.32×10^{-02} secs	3.60×10^{-07} secs	$2.22\times10^{+00}$ secs	4.35×10^{-07}
10^9	636619965	$1.00000005945416 \times 10^{+18}$	4.31×10^{-02} secs	4.01×10^{-07} secs	$1.15\times10^{+01}$ secs	8.61×10^{-07}

Table 4: A comparison of the results obtained by using the algorithm of this paper to evaluate prolate spheriodial wave functions of order 0 with those obtained via the Osipov-Rokhlin algorithm [22].