GRAVITATIONAL RADIATION OF A BINARY PULSAR Avigail Lithwick

ABSTRACT The orbiting pulsar PSR 1913+16 upholds the theory of general relativity based on observations made between the years 1974 and 1981. According to the general relativistic quadrupole formula, one should expect the orbital period derivative to have the value of $dP/dt = -2.4 \times 10^{-12}$. The aforementioned observations yielded a value of $dP/dt = -(2.30 \pm 0.22) \times 10^{-12}$. The outstanding agreement of the two would seem to confirm the veracity of the theory general relativity.

INTRODUCTION

The orbiting pulsar PSR 1913+16 was discovered in 1975 by Hulse and Taylor, and has since been observed in order to determine a number of its properties. Specifically, the observations yielded the period time of the orbit pulsar in a binary orbit with another star about a common center of mass.

The obtained data of the pulse arrival time were fit to a model in order to calculate the pulse phase. The observed parameters led to the calculation of a number of quantities, which may be categorized as follows

1. Traditional pulsar timing measurables. Parameters in this calegory include the period P, as well as its first and second time derivatives.

2. Classical derivable elements of a binary orbit. Examples of this include the eccentricity e, as well as the binary orbital period P_b .

3. Relativistic terms. An important parameter in this category is the rate of change of the orbital period dP_b/dt

4. A number of additional terms, such as the angle of the orbit i.

In this paper, the mathermatical basis for calculating a number of these parameters will be shown, which will lead to the validation of the theory of general relativity.

I GRAVITATIONAL WAVES IN A WEAK FIELD

In order to determine the change in energy in a system, we will first determine how the gravitational waves propagate. In particular, we will examine the gravitational waves in a weak gravitational field. Observing Einstein's equations in a vacuum, $T^{\mu\nu} = 0$,

$$(-\partial^2/\partial t^2 + \nabla^2)\overline{h}^{\alpha\beta} = 0(1)$$

where $\overline{h}^{\alpha\beta} = h^{\alpha\beta} - 1/2 \cdot \eta^{\alpha\beta} h$ is the trace reverse of $h_{\alpha\beta}$ We can rewrite(1) as

$$\eta^{\mu\nu}\overline{h}^{\alpha\beta}_{,\mu\nu} = 0(2)$$

If we say that (1) has a wavelike solution of the form:

$$\overline{h}^{\alpha\beta} = A^{\alpha\beta} exp(ik_{\alpha}x^{\alpha})(3)$$

where $A^{\alpha\beta}$ are constants of a tensor, and k_{α} are constants of a one form. Then we can rewrite (2) as:

$$\eta^{\mu\nu}k_{\mu}k_{\nu}\overline{h}^{\alpha\rho} = 0(4)$$

We require therefore that $\eta^{\mu\nu}k_{\mu}k_{\nu}=0$. This implies that the four vector k^{α} is the direction of travel of the wave, which moves at the speed of light. Imposing a gauge condition $\overline{h}_{,\beta}^{\alpha\beta} = 0$ onto the Einstein equations, we find

that

$$A^{\alpha\beta}k_{\beta} = 0(5)$$

The above shows that $A^{\alpha\beta}$ must therefore be orthogonal to k.

Let us refer to k^0 as the frequency of the wave (ω) . For a parameter λ , and a constant position vector at $\lambda = 0$, the photon then travels on a curve:

$$x^{\mu}(\lambda) = k^{\mu}\lambda + l^{\mu}$$

The solution $A^{\alpha\beta}exp(ik_{\mu}\mu)$ is called a plane wave. As a result, the combination of solutions for (1) and the gauge condition yield a superposition of plane waves.

Recalling the fact that an altered gauge remains a gauge within the Lorentz class, we will solve:

$$(-\partial^2/\partial t^2 + \nabla^2)\xi_\alpha = 0$$

With a solution of

$$\xi_{\alpha} = B_{\alpha} exp(ik_{\mu}x^{\mu})$$

Where k^{μ} is the same as before, and B_{α} is a constant. We will get a change in $h^{\alpha\beta}$, and a consequent change in $\overline{h}_{\alpha\beta}$. Namely,

$$h_{lphaeta} - > h_{lphaeta} - \xi_{lpha,eta} - \xi_{eta,lpha}$$

Dividing out the common exponential factor, and choosing appropriate B_{α} , we can impose the following restrictions:

$$A^{\alpha}_{\alpha} = 0(6a)$$
$$A_{\alpha\beta}U^{\beta} = 0(6b)$$

for some fixed four velocity U. These two equations, combined with equation (5), are the conditions for the Transverse-Traceless (TT) gauge.

II RESONANT DETECTOR

There are great technical difficulties involved in detecting gravitational waves. In order to better understand what is involved in their detection, we will consider an idealized resonant detector. Namely, let us consider two masses, m, connected by a massless spring, with spring constant k. The unstretched length of the spring is l_0 , and the damping constant is given by ν

Examining the system on the x-axis of the Transverse-Traceless(TT) coodinate system, and having the masses at positions x_1, x_2 , the masses obey the following:

$$mx_{1,00} = -k(x_1 - x_2 + l_0) - \nu(x_1 - x_2)_{,0}(2.1a);$$

$$mx_{2,00} = -k(x_2 - x_1 - l_0) - \nu(x_2 - x_1)_{,0}(2.1b)$$

For $\xi = x_2 - x_1 - l_0$; $\omega_0^2 = 2k/m$; $\gamma = \nu/m$ Combining with (2.1a) and (2.1b):

$$\xi_{,00} + 2\gamma\xi_{,0} + \omega_0^2\xi = 0(2.2)$$

This is the fundamental equation of the detector's response due to the gravitational wave, which is that of a simple damped harmonic oscillator. In the case of a pulsar or binary star, this system can be used as a resonant detector for gravitational waves with fixed frequencies.

If we say that

$$h_{xx}^{TT} = A\cos\Omega t$$

we can state that the steady solution to ξ will simply be

$$\xi = R\cos(\Omega t + \phi),$$

where R and ϕ are given by:

$$R = l_0 \Omega^2 A / 2((\omega_0 - \Omega)^2 + 4\Omega^2 \gamma^2)^{1/2}$$

$$tan\phi = 2\gamma\Omega/(\omega_0^2 - \Omega^2)$$

III APPROXIMATION OF WAVE GENERATION

Consider

$$(-\partial^2/\partial t^2 + \nabla^2)\overline{h}_{\mu\nu} = -16\pi T_{\mu\nu}(3.1)$$

In this section, we will try to find an approximate solution for this equation. Separating $T_{\mu\nu}$ into spatial and time-dependent functions, and assuming that the time-dependent oscillates with frequency Ω (the real part of):

$$T_{\mu\nu} = S_{\mu\nu}(x^i)exp(-i\Omega t)$$

this assumption is valid for two reasons: time dependence can generally be reduced to a sum over sinusoidal motions by Fourier analysis, and the fact that the binary system is periodic.

Another restriction to impose- the slow motion assumption- requires that the spatial part of $T_{\mu\nu}$ be much smaller than $2\pi/\Omega$. This implies that the typical velocity in the system be much less than 1.

Assuming that the solution has the form

$$\overline{h}_{\mu\nu} = B_{\mu\nu}(x^i)exp(-i\Omega t)(3.2),$$

and combining this with 3.1:

$$(\nabla^2 + \Omega^2) B_{\mu\nu} = -16\pi - S_{\mu\nu},$$

the amplitude of the gravitational waves will drop off as a function of 1/r, like all radiation fields. The simplest solution for $B_{\mu\nu}$ will have an exponential term that increases with r and an exponential term that decreases with r. However, the term that decreases with r can be discarded, as this represents a term travelling towards the origin, and we are interested in waves emitted from the origin. Thus, we can express:

$$B_{\mu\nu} = (A_{\mu\nu}exp(i\Omega r))/r$$

where $A_{\mu\nu}$ is a constant, to be determined in terms of the source.

Integrating each term of $B_{\mu\nu}$ over a sphere with radius $\epsilon \ll 2\pi/\Omega$, we obtain the following two terms:

$$\int \Omega^2 B_{\mu\nu} d^3x \le \Omega^2 |B_{\mu\nu}|_{max} 4\pi \epsilon^3 / 3(3.3)$$
$$\int \nabla^2 B_{\mu\nu} d^3x = \int n \cdot \nabla B_{\mu\nu} dS = -4\pi A_{\mu\nu}(3.4)$$

Note that the integral in the middle of equation 3.4 is a closed integral, by Gauss' Theorem. Since, in this case, we are looking at the source, we get the right hand side of equation 3.4. Calling $S_{\mu\nu}d^3x = J_{\mu\nu}$, and evaluating the combined (3.3)+(3.4) for ϵ -i0, we get:

$$A_{\mu\nu} = 4J_{\mu\nu}$$

Recalling that $\overline{h}_{\mu\nu}$ are related functions, as they are components of a tensor, we will find terms to simplify $\overline{h}_{\mu\nu}$.

In particular,

$$-i\Omega J^{\mu 0}e^{-i\Omega t} = \int T^{\mu\nu}_{,0}d^3x$$

since $T^{\mu\nu}$ obeys the conservation law, $T^{\mu 0}_{,0} = -T^{\mu k}_{,k}$ as a result. Therefore:

$$i\Omega J^{\mu 0}e^{-i\Omega t} = \int T^{\mu k}_{,k} d^3x$$

Applying Gauss' Theorem to a volume contained in the source, we see that $T^{\mu\nu} = 0$ on the surface. For a nonzero Ω , this implies that

$$J^{\mu 0} = \overline{h}^{\mu 0} = 0$$

Recall that

$$d^2/dt^2 \int T^{00} x^l x^m d^3 x = 2 \int T^{lm} d^3 x (3.5)$$

Additionally, for a source in slow motion, T^{00} is the Newtonian mass density, ρ . The integral on the left hand side of equation (3.5) is the quadrupole moment tensor of the mass distribution. It is denoted:

$$I^{lm} = \int T^{00} x^l x^m d^3 x = D^{lm} e^{-i\Omega t}$$

Neglecting all terms of order higher than r^{-2} , and terms nondominant in slow motion approximation, we obtain:

$$\overline{h}_{jk} = -2\Omega^2 D_{jk} e^{i\Omega(r-t)}/r$$

Additionally, if we use previous information about TT gauges, one that is transverse to the radial direction (where the measured point on the wave is travelling in the z direction), we see that:

$$\overline{h}_{xx}^{TT} = -\overline{h}_{yy}^{TT} = -\Omega^2 (\overline{I}_{xx} - \overline{I}_{yy}) e^{i\Omega(r-t)} / r(3.6)$$

Where \overline{I}_{ik} is called the reduced quadrupole moment tensor, and is:

$$\overline{I}_{jk} = i_{jk} - 1/3\delta_{jk}I_l^l$$

IV ENERGY DUE TO GRAVITATIONAL WAVES

Similar to gravitational waves, detectors can also add energy into objects that they pass through. It can further be understood that they can take energy away from their sources. This is a crucial point, as it will help us in understanding what happens precisely with orbiting binary stars.

We previously examined a single oscillator constructed of two masses, and assumed that this detector had a negligible effect on the gravitational wave field. Unfortunately, this is not completely correct. Continuing with previous reasoning, as the detector extracts energy from the waves, the waves become increasingly weaker after passing through the detector. In other words, the amplitude in the downstream will be lower than that in the upstream.

We are thus interested in the energy carried by a wave across a surface per unit area per unit time, which is just the energy flux. Rather than considering a single oscillator, it is more logical to think of a system with many oscillators on the z = 0 plane. Say we have σ oscillators per unit area, on a nearly continuous distribution of oscillators. In the TT gauge, the incident wave is thus:

$$\overline{h}_{xx}^{TT} = A cos \Omega(z-t) = -\overline{h}_{yy}^{TT}$$

Note that, due to the fact that the scattered energy due to friction can be compensated for by the work applied by the tidal gravitational forces of the wave onto the spring, each oscillator will have a steady oscillation. Recall from II, steady oscillation has the form:

$$\xi = Rcos(\Omega t + \phi)$$

where R and ϕ are given, as before:

$$R = l_0 \Omega^2 A / 2((\omega_0 - \Omega)^2 + 4\Omega^2 \gamma^2)^{1/2}$$
$$tan\phi = 2\gamma \Omega / (\omega_0^2 - \Omega^2)$$

The energy from the wave to each oscillator can be written as:

$$dE/dt = m\gamma (d\xi/dt)^2$$

Averaging the above expression over a period of $2\pi/\Omega$, which is the energy to each oscillator per unit time, gives:

$$\langle dE/dt \rangle = 1/2m\gamma\Omega^2 R^2.$$

Considering the decrease of the flux, F, on the z = 0 plane, with σ oscillators per unit area:

$$\delta F = -1/2\sigma m \gamma \Omega^2 R^2(4.1)$$

Consider the quadrupole tensor for each oscillator, where the amplitude is half the total stetching of the spring, R:

$$I_{xx} = ml_0 Rcost(\Omega t + \phi)$$

Note that the wave amplitude produced by each oscillator a distance r away, is much smaller than the incident wave. It is given by:

$$\delta \overline{h}_{xx} = -2\Omega^2 m l_0 R \cos(\Omega(r-t) - \phi)/r$$

where we ignored terms in the above two equations, as R is much smaller than l_0 , and 2Ω is negligible compared with Ω .

Consider a point, p_1 a distance z away from the plane of oscillators, centred at a point p_2 . Taking any oscillator, at a point p_3 , a distance $\overline{\omega}$ away from p_2 , we get that $r = ((\overline{\omega})^2 + z^2)^{1/2}$.

At p_1 , the field due to oscillators is:

$$\delta \overline{h}_{xx}^{total} = -2m\Omega^2 l_0 R 2\pi \int_0^\infty \sigma \cos(\Omega(r-t) - \phi) dr.$$

Observing the fact that the further away the observation point is, the smaller the impact, we can say that σ is proportional to $e^{(}-\epsilon r)$, in the limit when ϵ goes to zero after integration.

The result is a net wave, given by:

$$\delta \overline{h}_{xx}^{total} = 4\pi m \sigma \Omega l_0 Rsin(\Omega(z-t) - \phi).$$

We will now place this in the TT gauge, so as to add this to the incident wave:

$$\delta \overline{h}_{xx}^{TT} = -\delta \overline{h}_{yy}^{TT} = 2\pi m \sigma \Omega l_0 Rsin(\Omega(z-t) - \phi).$$

The net result of the incident wave and the wave amplitude due to each oscillator is:

$$\delta h' net)_{xx} = (A - 2m\sigma\pi\Omega l_0 Rsin(\Omega(z-t) - \psi))$$

where the phase shift ψ is given by:

$$tan\psi = -2(m\sigma\pi\Omega l_0Rcos\phi)/A$$

Hence, the net reduction in the amplitude, apart from the phase shift, shall be:

$$\delta A = -2m\pi\sigma\Omega l_0 Rsin\phi(4.2).$$

Dividing (4.1) by (4.2), and ordering to remove R and ϕ give the result of

$$\delta F/\delta A = \Omega^2 A/16\pi$$

which does not in the least depend on any properties of the oscillators. This is useful, as it allows us to generalize, and thus obtain:

$$F = \Omega^2 < \overline{h}_{\mu\nu}^{TT} \overline{h}^{TT\mu\nu} > (4.3)$$

where the average of the square of the wave is given by:

$$<(\overline{h}_{xx}^{TT})^2>=1/2A^2.$$

We will now examine the specific case of an isolated radiating system. Integrating (4.3) over a sphere, distance r along the z axis (z is the direction from the centre of coordinates where the radiation originates) gives:

$$F = \Omega^6 < 2\overline{I}_{ij}\overline{I}^{ij} - 4\overline{I}_{zj}\overline{I}^j_z + \overline{I}^2_{zz} > /16\pi r^2.$$

The total luminosity is the integral of F over the sphere with radius r. Therefore, the luminosity L of a source of gravitational waves is:

$$L = \int Fr^2 \sin\theta d\theta d\phi = 1/5\Omega^6 < \overline{I}_{ij}\overline{I}^{ij} > .$$

For a general time dependence, this yields:

$$L = 1/5 < \dot{\bar{I}}_{ij}\dot{\bar{I}}^{ij}$$

where the dots denote third time derivatives.

We are finally ready to consider our case, that of the binary pulsar. We will get the following for L:

$$L = 8/5m^2 l_0^4 \omega^6 = 4.0(m\omega)^{10/3}$$

The Newtonian energy, with orbital radius $r = 1/2l_0$ is given by:

$$E = M\omega^2 r^2 - M^2/2r$$

Comparing with (10.3), taking the logarithms, differentiating, and seeing that dE/dt = L, we obtain:

$$L/E = -2/31/PdP/dt.$$

This shows the change in the period. Note that the change in the period decreases. Before interpreting, it is important to state one last fact. The change

in energy and change in period depend upon the eccentricity of the orbit. The relativistic prediction of the change of orbit prediction is thus:

$$dP/dt = -2.4 \times 10^{-12}.$$

This agrees exceptionally well with the observed value of

$$dP/dt = -(2.30 \pm 0.22) \times 10^{-12}$$

CONCLUSION

The theory of general relativity successfully predicts the energy loss due to gravitational radiation in the case of the orbiting pulsar PSR 1913+16. Additionally, the theory of general relativity is the only theory which accurately predicts the energy loss. Thus, it may be concluded that the orbiting pulsar validates the theory of general relativity.