GRAVITY IN NONCOMMUTATIVE GEOMETRY

CHRIS GEORGE

Introduction

The traditional arena of geometry and topology is a set of points with some particular structure that, for want of a better name, we call a space. Thus, for instance, one studies curves and surfaces as subsets of an ambient Euclidean space. It was recognized early on, however, that even such a fundamental geometrical object as an elliptic curve is best studied not as a set of points (a torus) but rather by examining functions on this set, specifically the doubly periodic meromorphic functions. Weierstrass opened up a new approach to geometry by studying directly the collection of complex functions that satisfy an algebraic addition theorem, and derived the point set as a consequence. In probability theory, the set of outcomes of an experiment forms a measure space, and one may regard events as subsets of outcomes; but most of the information is obtained from "random variables", i.e., measurable functions on the space of outcomes.

In noncommutative geometry, under the in influence of quantum physics, this general idea of replacing sets of points by classes of functions is taken further. In many cases the set is completely determined by an algebra of functions, so one forgets about the set and obtains all information from the functions alone. Also, in many geometrical situations the associated set is very pathological, and a direct examination yields no useful information. The set of orbits of a group action, such as the rotation of a circle by multiples of an irrational angle, is of this type. In such cases, when we examine the matter from the algebraic point of view, we often obtain a perfectly good operator algebra that holds the information we need; however, this algebra is generally not commutative.

The Gelfand-Naimark theorem gives us a complete equivalence between the category of locally compact Hausdorff spaces with continuous maps and the category of commutative C^*-algebras with *-homomorphisms. In a famous paper [9] that has become a cornerstone of noncommutative geometry, Gelfand and Naimark in 1943 characterized the involutive algebras of operators by just dropping commutativity from the most natural axiomatization for the algebra of continuous functions on a locally compact Hausdorff space. Thus, a noncommutative C^*-algebra will be viewed as the algebra of continuous functions on some 'virtual noncommutative space. The algebra-topology duality can be neatly summed up with the following dictionary, from [15],

TOPOLOGY	ALGEBRA
Locally compact space	C^*-algebra
Compact space	Unital C^*-algebra
Compactification	Unitization
Continuous proper map	$*$-homomorphism
Homeomorphism	Automorphism
Open subset	Ideal
Closed subset	Quotient algebra
Second countable	Separable
Measure	Positive functional

In order to go further however, particularly with regard to physical examples, we need to move beyond this noncommutative topology, and into some sort of differential structure. This will be the focus of this paper. This new calculus can be described simply in the following dictionary. For an involutive algebra A, we construct a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{H}$ is the Hilbert space on which $\mathcal{A}$ is realized, and D is a selfadjoint unitary on $\mathcal{H}$. The dictionary then will be [4],

CLASSICAL	NONCOMMUTATIVE
Complex function	Operator on $\mathcal{H}$
Real function	Selfadjoint operator on $\mathcal{H}$
Infinitesimal	Compact operator
Differential df	Commutator $da = [D, a]$
Integral	Dixmier trace tr_ω

The first section of this paper will be devoted to developing this noncommutative differential geometry. In the second section, we will apply these new tools to construct a gravity model through noncommutative geometry.

1. The Spectral Calculus

In this section we will introduce the basic notions of the noncommutative generalization of the usual calculus on manifolds.

1.1. Infinitesimals.

Recall that for any T $\in \mathcal{K}(\mathcal{H})$, we have the polar decomposition $T= U\ |T|$, where $|T|= (T^*T)^{1/2}$. Then, the eigenvalues of $|T|$, with multiplicity, are called the characteristic values of T. These characteristic values, denoted by $\{\mu_n(T)\}_{n\in\mathbb{N}}$, are enumerated in decreasing order. So, $\mu_0(T) = \|T\|$, the operator norm of T, and $\mu_n(T) \to 0$ as $n \to \infty$.

Because compact operators are, in a sense, 'small', they play the role of infinitesimals in Connes' theory. The rate of decay of $\{\mu_n(T)\}$ as $n \to \infty$ tells us the size of the infinitesimal $T \in \mathcal{K}(\mathcal{H})$, as set out below:

Definition 1.1.1. For any $\alpha \in \mathbb{R}^+$, the infinitesimals of order α are all $T \in \mathcal{K}(\mathcal{H})$ such that

$$\mu_n(T) = O(n^{-\alpha}), \quad as \quad n \to \infty \tag{1.1.1}$$

$$ie.\ \exists\, C < \infty\ such\ that\ \mu_n(T) \leq Cn^{-\alpha}, \forall\, n \geq 1 \tag{1.1.2}$$

Now, for any two compact operators T_1 and T_2, we have the submultiplicative property[13],

$$\mu_{n+m}(T_1T_2) \leq \mu_n(T_1)\mu_m(T_2), \tag{1.1.3}$$

implying that the orders of infinitesimals are well-behaved,

$$T_i \ of \ order \ \alpha_i \ \Rightarrow \ T_1T_2 \ of \ order \ \alpha_1 + \alpha_2 \tag{1.1.4}$$

And lastly, infinitesimals of order α form a (two-sided) ideal in $\mathbf{B}(\mathcal{H})$, since for any $B \in \mathbf{B}(\mathcal{H})$ and $T \in \mathcal{K}(\mathcal{H})$ [13],

$$\mu_n(TB), \mu_n(BT) \leq \|B\|\mu_n(T) \tag{1.1.5}$$

1.2. **The Dixmier Trace.** In the noncommutative approach, as in the ordinary calculus, one looks for an 'integral' neglecting infinitesimals of order > 1. The Dixmier trace, constructed below, will be exactly what we are looking for.
For a positive compact operator T, we have

$$tr\ T = \sum_0^{\infty} \mu_n(T) \tag{1.2.1}$$

Now, in general, infinitesimals of order 1 are not trace class, since the only bound we have on the characteristic values is $\mu_n \leq C\frac{1}{n}$, for some positive constant C. But, from this we see that the usual trace (1.2.1), is at most logarithmically divergent for (positive infinitesimals order 1:

$$\sum_0^{N-1} \mu_n(T) \ \leq \ C\, logN \tag{1.2.2}$$

We wil use the Dixmier trace as a way to simply extract this coefficient C of logarithmic divergence. It is rather interesting that this coefficient behaves like a trace[8].

The ideal of compact operators which are infinitesimals of order 1 will be denoted by $\mathcal{L}^{(1,\infty)}$. For any positive $T \in \mathcal{L}^{(1,\infty)}$, the natural thing to do is take the limit of the cut-off sums,

$$\lim_{N\to\infty} \frac{1}{logN} \sum_0^{N-1} \mu_n(T) \tag{1.2.3}$$

However, problems with linearity and convergence arise in this definition. Now for any $T \in \mathcal{K}(\mathcal{H})$ consider the sums,

$$\sigma_N(T) \ = \ \sum_0^{N-1} \mu_n(T)\ , \ \gamma_N(T) \ = \ \frac{\sigma_N(T)}{logN} \tag{1.2.4}$$

Then, we have[4],

$$\sigma_N(T_1 \ + \ T_2) \ \leq \ \sigma_N(T_1) \ + \ \sigma_N(T_2), \ \forall\, T_1,\, T_2, \tag{1.2.5}$$

$$\sigma_{2N}(T_1 \ + \ T_2) \ \geq \ \sigma_N(T_1) \ + \ \sigma_N(T_2), \ \forall\, T_1,\, T_2 \ > \ 0 \tag{1.2.6}$$

As a result, for any two positive operators T_1 and T_2,

$$\gamma_N(T_1 + T_2) \le \gamma_N(T_1) + \gamma_N(T_2) \le \gamma_{2N}(T_1 + T_2)(1 + \frac{log2}{logN}) \tag{1.2.7}$$

Thus, convergence would imply linearity, but the sequence $\{\gamma_N\}$ (though bounded) does not, in general, converge. However in most examples of physical interest, $\{\gamma_N\}$ is, in fact, convergent. So, (1.2.3) certainly yields a positive linear functional, since $\mathcal{L}^{(1,\infty)}$ is generated by its positive elements. Thus, we will define the Dixmier trace, tr_ω, by

$$tr_\omega(T) = \lim_{N\to\infty} \frac{1}{logN} \sum_{0}^{N-1} \mu_n(T) \tag{1.2.8}$$

Also, from (1.1.5), for any $T \in \mathcal{L}^{(1,\infty)}$,

$$tr_\omega(BT) = tr_\omega(TB), \ \forall\, B \in \mathbf{B}(\mathcal{H}) \tag{1.2.9}$$

And, any infinitesimal T of order greater than 1 satisfies

$$\mu_n(T) = o(\frac{1}{n}), \quad i.e.\ n\mu_n(T) \to 0\ as\, n \to \infty \tag{1.2.10}$$

So the corresponding sequence $\gamma_N(T)$ converges to zero, showing that the Dixmier trace vanishes on infinitesimals of order > 1. Thus, the Dixmier trace is, in fact, a trace whose domain is $\mathcal{L}^{(1,\infty)}$, and neglects all infinitesimals of order > 1.

1.3. **Spectral Triples.**

Here we will introduce the main concept used by Connes to develop the analogue of differential calculus for noncommutative algebras.

Definition 1.3.1. A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive algebra $\mathcal{A}$ of bounded operators on the Hilbert space $\mathcal{H}$, together with a self-adjoint operator D on $\mathcal{H}$ (called the Dirac operator) satisfying:

- The resolvent $(D - \lambda)^{-1}$, $\lambda \in \mathbb{C}\backslash\mathbb{R}$, is a compact operator on $\mathcal{H}$;
- The commutator $[D, a] = Da - aD \in \mathbf{B}(\mathcal{H})$, $\forall\, a \in \mathcal{A}$

These conditions imply that the collection $\{\lambda_n\}$ of eigenvalues of D forms a discrete subset of $\mathbb{R}$ [4]. In addition, $(D - \lambda)^{-1}$ being compact has characteristic values $\mu_n((D - \lambda)^{-1}) \to 0$, and thus $|\lambda_n| = \mu_n(|D|) \to \infty$.

Also, consider the derivation δ on $\mathbf{B}(\mathcal{H})$ defined by

$$\delta(T) = [|D|, T]. \tag{1.3.1}$$

Given this derivation δ, the subalgebra $\mathcal{A}^k \subset \mathcal{A}$, with $k \ge 2$, consists os all elements $a \in \mathcal{A}$ such that both a, and $[D, a]$ are in the domain of δ^{k-1}. The elements of $\bigcap \mathcal{A}_k$ are said to be of class C^∞.

Now consider a closed n-dimensional Riemannian spin manifold (M, g). The corresponding spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called the *canonical triple* over M. In this example we have,

- $\mathcal{A} = \mathcal{F}(M)$, the algebra of smooth, complex-valued functions on M.
- $\mathcal{H} = L^2(M, S)$, the space of square integrable sections of the irreducible spinor bundle over M, with the natural scalar product.
- D is the Dirac operator associated with the Levi-Cevita connection of the metric g.

Recall that taking the norm closure of an involutive algebra $\mathcal{A}$, yields a C^*-algebra $\overline{\mathcal{A}}$. Leaving the details to [10], we state the following proposition without proof.

Proposition 1.3.2. *For the canonical triple $(\mathcal{A}, \mathcal{H}, D)$, we have*

- *M is the structure space of the C^*-algebra $\overline{\mathcal{A}}$ of continuous functions on M (This, of course, is simply the Gelfand-Naimark theorem).*
- *The geodesic distance between any two points $p, q \in M$ is given by*

$$d(p,q) = \sup_{f \in \mathcal{A}} \{|f(p) - f(q)|; \|[D,f]\| \leq 1\}, \ \forall p, q \in M. \tag{1.3.2}$$

- *The Riemannian measure on M is given by*

$$\int_M f \ = \ C \, tr_\omega(f|D|^{-n}), \ \forall \, f \in \mathcal{A}, \tag{1.3.3}$$

Where $C = 2^{(n-[n/2]-1)}\pi^{n/2} n\Gamma(n/2)$.

In accordance with this proposition we can define noncommutative analogues of (1.3.2), and (1.3.3). The distance function on the state space $\mathcal{S}(\overline{\mathcal{A}})$ is given by

$$d(\phi, \xi) = \sup_{a \in \mathcal{A}} \{|\phi(a) - xi(a)|; \|[D,a]\| \leq 1\}, \ \forall \phi, \xi \in \mathcal{S}(\overline{\mathcal{A}}). \tag{1.3.4}$$

To define the analogue of the measure integral, we first need the notion of the dimension of a spectral triple.

Definition 1.3.3. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is of dimension n > 0 if $|D|^{-n}$ is an infinitesimal of order 1.

Now, given an n-dimensional spectral triple, for any $a \in \mathcal{A}$ its integral is defined by

$$\int a = \frac{1}{V} tr_\omega a|D|^{-n}, \tag{1.3.5}$$

where the constant V is determined by the characteristic values of $|D|^{-n}$. More precisely, $\mu_j(|D|^{-n}) \leq \frac{V}{j}$ for $j \to \infty$.

1.4. **Universal Differential Forms.**

For a unital involutive algebra $\mathcal{A}$ over $\mathbb{C}$, the universal differential algebra of forms $\Omega\mathcal{A} = \bigoplus_p \Omega^p\mathcal{A}$ is a graded algebra which we will construct below. In degree 0, we have $\Omega^0\mathcal{A} = \mathcal{A}$. The space of one-forms, $\Omega^1\mathcal{A}$, is generated as a left $\mathcal{A}$-module by symbols of degree δa, for $a \in \mathcal{A}$, satisfying the relations

$$\delta(ab) = (\delta a)b + a\delta b \tag{1.4.1}$$

$$\delta(\alpha a + \beta b) = \alpha\delta a + \beta\delta b \tag{1.4.2}$$

The Leibniz rule (1.4.1) automatically gives

$$\delta 1 = \delta(1 \cdot 1) = (\delta 1) \cdot 1 + 1 \cdot (\delta 1) = 2(\delta 1) \implies \delta 1 = 0 \implies \delta\mathbb{C} = 0.$$

So, a generic $\omega \in \Omega^1\mathcal{A}$ is just a finite linear combination

$$\omega = \sum_{finite} a_i \delta b_i, \quad for\ some\ a_i, b_i \in \mathcal{A} \tag{1.4.3}$$

We can also give $\Omega^1\mathcal{A}$ a right $\mathcal{A}$-module structure by setting

$$\Big(\sum a_i\delta b_i\Big)c = \sum a_i(\delta b_i)c = \sum a_i\delta(b_ic) - \sum a_ib_i\delta c \tag{1.4.4}$$

where we have used the Leibniz rule to obtain the final equality.
The map

$$\delta : \mathcal{A} \to \Omega^1\mathcal{A}$$

can be considered a derivation of $\mathcal{A}$ with values in the bimodule $\Omega^1\mathcal{A}$. Then, the pair $(\delta, \Omega^1\mathcal{A})$ is characterized by the universal property[4].

In higher degrees, the space $\Omega^p\mathcal{A}$ is given by

$$\Omega^p\mathcal{A} = \underbrace{\Omega^1\mathcal{A}\Omega^1\mathcal{A}\cdots\Omega^1\mathcal{A}}_{p-times} \tag{1.4.5}$$

where the product of any two one-forms is defined simply by 'juxtaposition',

$$(a_0\delta a_1)(b_0\delta b_1) = a_0(\delta a_1)b_0\delta b_1) = a_0\delta(a_1b_0)\delta b_1 - a_0a_1\delta b_0\delta b_1,\ \forall a_i, b_i \in \mathcal{A}. \tag{1.4.6}$$

Thus, $\Omega^p\mathcal{A}$ consists of finite linear combinations of monomials of the form

$$\omega = a_0\delta a_1\delta a_2\cdots\delta a_p,\ a_i \in \mathcal{A}. \tag{1.4.7}$$

So, the product of a p-form and a q-form yields a $(p+q)$-form which is, again, defined by juxtaposition, and rearranging the result using the Leibniz rule,

$$\begin{aligned}
&(a_0\delta a_1\cdots\delta a_p)(a_{p+1}\delta a_{p+2}\cdots\delta a_{p+q})\\
&= a_0\delta a_1\cdots(\delta a_p)a_{p+1}\delta a_{p+2}\cdots\delta a_{p+q}\\
&= (-1)^p a_0a_1\delta a_2\cdots\delta a_{p+q}\\
&\quad + \sum_{i=1}^{p}(-1)^{p-i}a_0\delta a_1\cdots\delta a_{i-1}\delta(a_ia_{i+1})\delta a_{i+2}\cdots\delta a_{p+q}
\end{aligned} \tag{1.4.8}$$

As with $\Omega^1\mathcal{A}$, $\Omega^p\mathcal{A}$ is a left $\mathcal{A}$-module by construction. It can also be endowed with a right $\mathcal{A}$-module structure in the same manner, using the Liebniz rule,

$$\begin{aligned}
(a_0\delta a_1\cdots\delta a_p)b &= a_0\delta a_1\cdots(\delta a_p)b\\
&= (-1)^p a_0a_1\delta a_2\cdots\delta a_p\delta b\\
&\quad + \sum_{i=1}^{p}(-1)^{p-i}a_0\delta a_1\cdots\delta a_{i-1}\delta(a_ia_{i+1})\delta a_{i+2}\cdots\delta a_p\delta b\\
&\quad + a_0\delta a_1\cdots\delta a_{p-1}(\delta a_p b),\ \forall a_i, b \in \mathcal{A}
\end{aligned} \tag{1.4.9}$$

Now extending δ so that for any p we have a map

$$\delta : \Omega^p \to \Omega^{p+1}$$

defined by the equation

$$\delta(a_0\delta a_1\cdots\delta a_p) = \delta a_0\delta a_1\cdots\delta a_p. \tag{1.4.10}$$

Then we have the relations

- $\delta^2 = 0$,
- $\delta(\omega_1\omega_2) = \delta(\omega_1)\omega_2 + (-1)^p\omega_1\delta\omega_2$, $\forall\omega_1 \in \Omega^p\mathcal{A}$, $\omega_2 \in \Omega\mathcal{A}$.

And lastly, we define an involution on $\Omega\mathcal{A}$ by

$$(\delta a)^* = -\delta a^*,\ \forall a \in \mathcal{A} \tag{1.4.11}$$

$$\begin{aligned}(a_0\delta a_1 \cdots \delta a_p)^* &= (\delta a_p)^* \cdots (\delta a_1)^* a_0^* \\ &= a_p^* \delta a_{p-1}^* \cdots \delta a_0^* \\ &+ \sum_{i=0}^{p-1} (-1)^{p+i} \delta a_p^* \cdots \delta(a_{i+1}^* a_i^*) \cdots \delta a_0^*.\end{aligned} \tag{1.4.12}$$

Thus, $(\Omega\mathcal{A}, \delta)$ is a graded differential involutive algebra, once again characterized by the universal property[4].

1.5. Connes' Differential Forms.

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, using a representation of the universal algebra $\Omega\mathcal{A}$ on $\mathbf{B}(\mathcal{H})$, we will construct the exterior algebra of forms. The map $\pi : \Omega\mathcal{A} \to \mathbf{B}(\mathcal{H})$ defined by

$$\pi(a_0\delta a_1 \cdots \delta a_p) = a_0[D, a_1] \cdots [D, a_p],\ a_i \in \mathcal{A} \tag{1.5.1}$$

is clearly a homomorphism, since both δ and $[d, \cdot]$ are derivations on $\mathcal{A}$. Also, it is a *-homomorphism, because $[D, a]^* = -[D, a^*]$.

The natural thing do next would be to define $\pi(\Omega)$ as the space of forms. In general, however, $\pi(\Omega) = 0$ does not imply $\pi(\delta\omega) = 0$. In order to proceed in constructing a true differential algebra, we will need to dispose of these so called junk forms.

Proposition 1.5.1. *Let $J_0 = \oplus_p J_0^p$ be the graded (two-sided) ideal of $\Omega\mathcal{A}$ given by*

$$J_0^p = \{\omega \in \Omega^p\mathcal{A},\ \pi(\omega) = 0\}. \tag{1.5.2}$$

Then, $J = J_0 + \delta J_0$ is a graded (two-sided) ideal of $\Omega\mathcal{A}$.

Proof. J is obviously graded, and the property $\delta^2 = 0$ implies it is differential. Let $\omega \in J$. Then, $\omega = \omega_1 + \delta\omega_2$, for some $\omega_1 \in \mathcal{J}_0^p$ and $\omega_2 \in J_0^{p-1}$. Then for any $\eta \in \Omega\mathcal{A}$, we have $\omega\eta = \omega_1\eta + \delta(\omega_2\eta) + (-1)^p\omega_2\delta\eta = \Big(\omega_1\eta + (-1)^p\omega_2\delta\eta\Big) + \delta(\omega_2\eta) \in J$. In a similar way, we find $\eta\omega \in J$. □

Definition 1.5.2. The graded differential algebra of Connes' forms over $\mathcal{A}$ is defined by

$$\Omega_D\mathcal{A} = \Omega\mathcal{A}/J \simeq \pi(\Omega\mathcal{A})/\pi(\delta J_0), \tag{1.5.3}$$

with the space of p-forms given by

$$\Omega_D^p\mathcal{A} = \Omega^p\mathcal{A}/J^p. \tag{1.5.4}$$

Now, since J is a differential ideal, the exterior differential δ defines a differential on $\Omega_D\mathcal{A}$, $d : \Omega_D^p\mathcal{A} \to \Omega_D^{p+1}$, defined by

$$d[\omega] = [\delta\omega] \simeq [\pi(\delta\omega] \tag{1.5.5}$$

with $\omega \in \Omega^p\mathcal{A}$ and $[\omega]$ the corresponding equivalence class in $\Omega_D^p\mathcal{A}$.

1.6. **Scalar Product of Forms.** While we showed that the Dixmier trace was indeed a trace state on $\mathcal{A}$, we need to extend the result to all of $\pi(\Omega\mathcal{A})$. This is not possible, in general, and some conditions need be imposed on the algebra $\mathcal{A}$. Recall that the subalgebra $\mathcal{A}^2 \subset \mathcal{A}$ is generated by all elements $a \in \mathcal{A}$ such that both a and $[D, a]$ are in the domain of the derivation δ defined in (1.3.1). It has been shown in [3] that provided $\mathcal{A}^2 = \mathcal{A}$ (or $\mathcal{A}^2$ is a sufficiently large subalgebra), the Dixmier trace is indeed a trace state on $\pi(\Omega\mathcal{A})$. As a result, the following three traces coincide, and will be taken as the definition of an inner product on $\pi(\Omega^p\mathcal{A})$,

$$\begin{aligned}\langle T_1, T_2\rangle_p &= tr_\omega(T_1^* T_2 |D|^{-n}) \\ &= tr_\omega(T_1^* |D|^{-n} T_2) \\ &= tr_\omega(T_2 |D|^{-n} T_1^*),\ \forall T_1, T_2 \in \pi(\Omega^p\mathcal{A}).\end{aligned} \tag{1.6.1}$$

And forms of different degrees are defined to be orthogonal.

Denote the completion of $\pi(\Omega^p\mathcal{A})$ by $\tilde{\mathcal{H}}_p$. Let P_p be the orthogonal projection of $\tilde{\mathcal{H}}_p$, with respect to the inner product (1.6.1), which projects onto the orthogonal complement of $\pi(\delta(J_0 \cap \Omega^{p-1}\mathcal{A}))$.

1.7. **Universal Connections on Modules.** In this section, we will develop the notion of a connection on a (finite projective), with respect to the universal calculus $\Omega\mathcal{A}$. Since $\Omega\mathcal{A}$ is the prototype for any calculus on $\mathcal{A}$, by a connection we really mean a *universal* connection, but this adjective will be dropped when there is no confusion.

Definition 1.7.1. A (universal) connection on the right $\mathcal{A}$-module $\mathcal{E}$ is a complex-linear map

$$\nabla : \mathcal{E} \otimes_\mathcal{A} \Omega^p\mathcal{A} \to \mathcal{E} \otimes_\mathcal{A} \Omega^{p+1}\mathcal{A}, \tag{1.7.1}$$

defined for any $p \geq 0$, satisfying the Leibniz rule

$$\nabla(\omega\rho) = (\nabla\omega)\rho + (-1)^p\omega\delta\rho,\ \forall\omega \in \mathcal{E} \otimes_\mathcal{A} \Omega^p\mathcal{A},\ \rho \in \Omega\mathcal{A}. \tag{1.7.2}$$

Thus, a connection is completely determined by its restriction $\nabla : \mathcal{E} \to \mathcal{E}\otimes_\mathcal{A}\Omega^1\mathcal{A}$, which satisfies

$$\nabla(\eta a) = (\nabla\omega)a + \eta \otimes_\mathcal{A} \delta a,\ a \in \mathcal{A}, \tag{1.7.3}$$

and then extends by using the Leibniz rule (1.7.2).

Proposition 1.7.2. *The map*

$$\nabla^2 = \nabla \circ \nabla : \mathcal{E} \otimes_\mathcal{A} \Omega^p\mathcal{A} \to \mathcal{E} \otimes_\mathcal{A} \Omega^{p+2}\mathcal{A}, \tag{1.7.4}$$

is $\Omega\mathcal{A}$-linear.

Proof. From the Leibniz rule 1.7.2, we have

$$\begin{aligned}\nabla^2(\omega\rho) &= \nabla\Big((\nabla\omega)\rho + (-1)^p\omega\delta\rho\Big) \\ &= (\nabla^2\omega)\rho + (-1)^{p+1}(\nabla\omega)\delta\rho + (-1)^p(\nabla\omega)\delta\rho + \omega\delta^2\rho \\ &= (\nabla^2\omega)\rho\end{aligned} \tag{1.7.5}$$

□

Definition 1.7.3. The restriction θ, of ∇^2 to $\mathcal{E}$,

$$\theta : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^2\mathcal{A}, \tag{1.7.6}$$

is the *curvature* of the connection.

The curvature is $\mathcal{A}$-linear and satisfies

$$\nabla^2(\eta \otimes_A \rho) = \theta(\eta)\rho,\ \forall \eta \in \mathcal{E},\ \rho \in \Omega\mathcal{A}. \tag{1.7.7}$$

Proposition 1.7.4. *The curvature θ satisfies the Bianchi identity,*

$$[\nabla, \theta] = 0. \tag{1.7.8}$$

Proof. Since $\theta : \mathcal{E} \to \Omega^2\mathcal{A}$, the map $[\nabla, \theta]$ makes sense. And,

$$[\nabla, \theta] = \nabla \circ \nabla^2 - \nabla^2 \circ \nabla = \nabla^3 - \nabla^3 = 0. \tag{1.7.9}$$

□

The next proposition, for which the proof can be found in [7], takes care of any concerns over the existence of connections.

Proposition 1.7.5. *There exists a connection on a right module if and only if it is projective.*

And, of course, we will only be considering finite projective modules in physical examples.

2. Gravity Models

While there have been a number of different approaches to constructing gravity models in noncommutative geometry (see, for instance [5], [11, 12]), we will follow the approach developed in [1, 2] for which our discussion of connections leads directly into.

2.1. **Connections Revisited.** Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with the associated differential calculus $(\Omega_D\mathcal{A}, d)$, then by Serre-Swan [14], the space $\Omega^1_D\mathcal{A}$ is the noncommutative analogue of the space of sections of the cotangent bundle. It is naturally a right $\mathcal{A}$-module, and we will assume it is projective of finite type, as well. In order to develop 'noncommutative Riemannian geometry' here, we will need an analogue of a metric on $\Omega^1_D\mathcal{A}$. There is a canonical Hermitian structure on $\Omega^1_D\mathcal{A}$ which is uniquely determined by the triple $(\mathcal{A}, \mathcal{H}, D)$ given by,

$$\langle \alpha, \beta \rangle_D = P_0(\alpha^*\beta) \in \mathcal{A},\ \alpha, \beta \in \Omega^1_D\mathcal{A}, \tag{2.1.1}$$

where P_0 is the orthogonal projector ont $\mathcal{A}$ determined by the scalar product (1.6.1. We will also assume that if $(\Omega^1_D\mathcal{A})'$ is the dual module, the Riemannian structure defines a right-module isomorphism,

$$\Omega^1_D\mathcal{A} \to (\Omega^1_D\mathcal{A})',\ \alpha \mapsto \langle \alpha, \cdot \rangle. \tag{2.1.2}$$

We can now define a linear connection which is formally the same as that in the remarks surrounding (1.7.3), taking $\mathcal{E} = \Omega^1_D\mathcal{A}$.

Definition 2.1.1. A linear connection on $\Omega^1_D\mathcal{A}$ is a complex-linear map

$$\nabla : \Omega^1_D \to \Omega^1_D \otimes_{\mathcal{A}} \Omega^1_D\mathcal{A}, \tag{2.1.3}$$

satisfying the Leibniz rule

$$\nabla(\alpha a) = (\nabla\alpha)a + \alpha da,\ \forall\alpha \in \Omega^1_D\mathcal{A},\ a \in \mathcal{A} \tag{2.1.4}$$

Definition 2.1.2. The Riemannian curvature of ∇ is the $\mathcal{A}$-linear map R_∇,

$$R_\nabla = \nabla^2 : \Omega^1_D \to \Omega^1_D\mathcal{A} \oplus_{\mathcal{A}} \Omega^1_D\mathcal{A}. \tag{2.1.5}$$

We call the connection ∇ *metric* if it is compatible with the Riemannian structure $\langle\cdot,\cdot\rangle_D$:

$$-\langle\nabla\alpha, \beta\rangle_D + \langle\alpha, \nabla\beta\rangle_D = d\langle\alpha, \beta\rangle_D,\ \forall\alpha, \beta \in \Omega^1_D\mathcal{A}. \tag{2.1.6}$$

Definition 2.1.3. The *torsion* of the connection ∇ is the map

$$T_\nabla : \Omega^1_D\mathcal{A} \to \Omega^2_D\mathcal{A},$$

$$T_\nabla = d - m \circ \nabla, \tag{2.1.7}$$

where m is just the multiplication operator, $m(\alpha \otimes_{\mathcal{A}} \beta) = \alpha\beta$.

It is easy to check that T_∇ is a tensor, and for an ordinary manifold with a linear connection, the above definition gives the cotangent (dual) space version of the usual definition of torsion.

Definition 2.1.4. A connection ∇ on $\Omega^1_D\mathcal{A}$ is a *Levi-Cevita* connection if it is metric, and its torsion vanishes.

Unlike in ordinary differential geoemtry, Levi-Cevita connections do not necessarily exist, or may not be unique for a given spectral triple. Now, for simplicity we will take $\Omega^1_D\mathcal{A}$ to be a free module with orthonormal basis $\{E^A, A = 1, \cdots, N\}$ with respect to the Riemannian structure $\langle\cdot,\cdot\rangle_D$,

$$\langle E^A, E^B\rangle_D = \eta^{AB} = diag(\delta_{AB}, \cdots, \delta_{AB}),\ A, B = 1, \cdots, N \tag{2.1.8}$$

As we noted earlier, a connection ∇ on $\Omega^1_D\mathcal{A}$ is completely determined by its action on 1-forms $\Omega^B_A \in \Omega^1_D\mathcal{A}$ defined by,

$$\nabla E^A = E^B \otimes_{\mathcal{A}} \Omega^A_B, A = 1, \cdots, N \tag{2.1.9}$$

Then the components of torsion $T^A \in \Omega^2_D\mathcal{A}$ and the curvature $R^B_A \in \Omega^2_D\mathcal{A}$ are given by

$$T_\nabla(E^A) = T^A$$

$$R_\nabla(E^A) = E^B \otimes_{\mathcal{A}} R^B_A,\ A = 1, \cdots, N \tag{2.1.10}$$

Then, by using (2.1.5), and (2.1.7) we obtain the *Cartan structure equations*,

$$T^A = dE^A - E^B\Omega^A_B,\ A = 1, \cdots, N$$

$$R^B_A = d\Omega^B_A + \Omega^C_A\Omega^B_C,\ A, B = 1, \cdots, N. \tag{2.1.11}$$

Now, the metric condition reads,

$$-\Omega^{A*}_C\eta^{AB} + \eta^{AC}\Omega^B_C = 0. \tag{2.1.12}$$

Since metricity and vanishing torsion don't necessarily fix the connection uniquely, sometimes additional constraints are imposed by requiring that the connection is Hermitian on 1-forms,

$$\Omega_A^B = \Omega_A^{B*} \tag{2.1.13}$$

The components of torsion and curvature transform in the expected way under a change orthonormal basis. For another orthonormal basis $\{\tilde{E}^A, A = 1, \cdots, N\}$ of $\Omega_D^1\mathcal{A}$, the relationship between the two bases is given by,

$$\tilde{E}^A = E^B(M^{-1})_B^A,\ E^A = \tilde{E}^B M_B^A \tag{2.1.14}$$

with

$$M_A^C(M^{-1})_C^B = (M^{-1})_A^C M_C^B = \delta_{AB} \tag{2.1.15}$$

ie. the matrix $M = (M_A^B)$ is invertible with inverse $M^{-1} = ((M^{-1})_A^B)$. The requirement that the new basis be orthonormal gives,

$$\begin{aligned}
\eta^{AB} &= \langle E^A, E^B\rangle_D \\
&= \langle \tilde{E}^C M_C^A, \tilde{E}^D M_D^B\rangle_D \\
&= (M_C^A)^*\langle \tilde{E}^C, \tilde{E}^D\rangle_D M_D^B \\
&= (M_C^A)^*\eta^{CD} M_D^B.
\end{aligned} \tag{2.1.16}$$

And so we get,

$$(M^{-1})_A^B = \eta^{AD}(M_C^D)\eta^{CB}, \tag{2.1.17}$$

that is, $M^{-1} = M^*$, and thus M is unitary.
It is easy to then find the components of curvature and torsion after an orthonormal change of basis,

$$\begin{aligned}
\tilde{\Omega_A}^B &= M_A^C\Omega_C^D(M^{-1})_D^B + M_A^C d(M^{-1})_C^B, \\
\tilde{R}_A^B &= M_A^C R_C^D(M^{-1})_D^B, \\
\tilde{T}^A &= T^B(M^{-1})_B^A.
\end{aligned} \tag{2.1.18}$$

Now, let $\{\epsilon_A, A = 1, \cdots, N\}$ be the dual basis of $\{E^A\}$. That is, $\epsilon_A \in (\Omega_D^1)'$, and

$$\epsilon_A(E^B) = \delta_{AB} \tag{2.1.19}$$

Then, by the isomorphism (2.1.2), for each ϵ_A, there is an $\hat{\epsilon}_A \in \Omega_D^1\mathcal{A}$ given by,

$$\epsilon_A = \langle\hat{\epsilon}_A, \alpha\rangle_D,\ \forall\alpha \in \Omega_D^1\mathcal{A},\ A = 1, \cdots, N \tag{2.1.20}$$

And thus,

$$\hat{\epsilon}_A = E^B\eta^{AB},\ A = 1, \cdots, N. \tag{2.1.21}$$

So, under an orthonormal change of basis, they transform as

$$\tilde{\hat{\epsilon}}_A = \hat{\epsilon}_B(M_A^B)^*,\ A = 1, \cdots, N. \tag{2.1.22}$$

Now, we are equipped to define the final noncommutative analogues needed for differential geometry.

Definition 2.1.5. The Ricci 1-forms, R_A^∇ of a connection ∇ are given by

$$R_A^\nabla = P_1(R_A^B(\hat{\epsilon}_B)^*) \in \Omega_D^1\mathcal{A} \tag{2.1.23}$$

Definition 2.1.6. The scalar curvature, r_∇, of a connection ∇ is defined by

$$r_\nabla = P_0(E^A P_1(R_A^B \hat{\epsilon}_B)^*) \tag{2.1.24}$$

where, again, the projectors P_0 and P_1 are those determined from the scalar product of forms (1.6.1).

One can easily check that the scalar curvature does not depend on the choice of orthonormal basis.

At long last, we have developed all the tools necessary for a noncommutative differential geometry upon which gravity may based. The culmination of this work is the noncommutative analogue of the *Einstein-Hilbert* action, defined by [2]

$$I_{HE}(\nabla) = tr_\omega r_\nabla |D|^{-n}. \tag{2.1.25}$$

Conclusion

In this paper we have developed a noncommutative analogue of Riemannian geometry. For a manifold M, with its canonical spectral triple, the classical and noncommutative geometries agree. While this is interesting in and of its own right, there are far greater implications. In [6], Connes and Lott modelled space-time by the so-called Connes-Lott space $M \times Y$, the product of a four-dimensional spin manifold M with a discrete internal space Y consisting of two points. Using the simply the noncommutative geometry of $M \times Y$, Connes and Lott derived a Lagrangian reproducing the Standard Model. Applying the techniques developed in this paper to the Connes-Lott space, as in [2], gives us an exciting, alternative method for bringing about the unification of gravity with quantum mechanics.

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