Introduction to Numerical Relativity

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1 Introduction

Einstein's theory of General Relativity is one of the most powerful physical theories ever written. It provides a complete description of space-time for any possible matter field. Unfortunately is blocked by the wall of complexity which are the Einstein Field Equations. The field equations are a complicated coupled system of highly nonlinear partial differential equations. In-spite of over 80 years of research, no one has been able to solve the field equations except for the most unphysically simplistic scenarios.

Solutions such as the Schwarzschild and the Kerr solution are interesting, but ultimately not very useful for predicting observations of observable astrophysical events such as the collision of two black holes. The field of numerical relativity is an effort to use computer technology obtain relativistic predictions of interesting phenomena.

In this paper I seek to introduce the fundamentals of the techniques of Numerical Relativity. Even with the aid of computers, the field equations pose a great challenge. As a result, I will have to introduce many simplifications in order to keep the length and complexity of the material within the scope of this paper. However, the derivation presented here will provide the reader with a basic understanding of the fundamental techniques of Numerical Relativity.

The bulk of this paper is spent in deriving (with some simplifications) a reformulation of the Einstein Field Equations useful for numerical analysis. It is this derivation that constitutes an introduction to the field of Numerical Relativity. A simple example is then included to help settle some of the concepts presented.

Then I conclude with a glance of current research and the international effort to solve the Einstein Field Equations in their full generality.

In this paper I use the standard convention found in most textbooks and papers on General Relativity. Greek indices (μ, ν, etc) run from 0 to 3, indicating all space-time coordinates, and Latin indices (i, j, etc) run from 1 to 3, indicating space coordinates.

2 Reformulation of the Field Equations

The first step in numerical relativity is to reformulate the Einstein Field Equations in a form suitable for numerical analysis. Although I mention the ADM 3+1 decomposition in passing, this section will make emphasis on a more recent approach. That is, to write field equations as a first order hyperbolic system. This approach has the benefit that all the procedures already developed for the computation of fluid dynamics can be applied to the field equations. For this derivation I will first have to introduce harmonic coordinates and harmonic slicing. Although recent developments permit us to circumvent harmonic slicing, the mathematical complexity involved puts them beyond the scope of this paper.

2.1 3+1 Decomposition

Introduced in 1962 by Arnowitt, Deser and Misner (ADM), the 3+1 formalism consists of separating the time coordinate from the 3 space coordinates. Then we can view the 4-metric as a 3-dimensional hyper-surface which evolves through time. We start by writing out the space-time distance between events as:

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -\alpha^2 dt + \gamma_{ij}(dx^i + \beta^i)(dx^j + \beta^j dt) \end{aligned} \tag{1}$$

Where α is called the *lapse function* and it represents the time measured by an observer moving perpendicular to the 3-dimensional hyper-surfaces. In other words, $d\tau = \alpha dt$ is the proper time between the 3-volume t and the 3-volume dt . If one decomposes the Ricci tensor into time-like and space-like components, and applying the vacuum field equations one can obtain four constraint equations and twelve evolution equations. The ADM 3+1 decomposition is the traditional approach for solving Einstein's Field Equations numerically.

2.2 Harmonic Coordinate Conditions

2.2.1 The Need for Coordinate Conditions

The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R$ contains 10 algebraically independent components. However, the Binacci identities $\nabla_\mu G^\mu_\nu = 0$ reduces the number of equations down to 6. Therefore, the Einstein Equations cannot determine a metric uniquely. This is where the choice of coordinates comes in. A particular coordinate system can be chosen (gauge choice), expressed in 4 coordinate conditions. These coordinate conditions complete the system of equations to permit the existence of a unique solution.

2.2.2 Harmonic Coordinate Conditions

In order to derive a hyperbolic formulation of the Einstein Field Equations, I will use the *harmonic coordinate condition*, which is defined by:

$$\Gamma^\lambda \equiv g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0 \tag{2}$$

The reason for the name *harmonic* coordinate will be made clear later on. To start, we will rewrite equation 7 in a more useful form. First of all, the metric $g_{\mu\nu}$ can be thought of as a 4×4 matrix. We then define g to be its determinant:

$$g = det(g_{\mu\nu}) \tag{3}$$

Using the formula for the inverse of a matrix (in this case $g^{\mu\nu}$) we can obtain[1]:

$$g^{\nu\alpha}\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} = \frac{1}{g}\frac{\partial g}{\partial x^{\mu}} \tag{4}$$

With this we can derive:

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^{\mu}} = \frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial x^{\mu}} = \frac{1}{2}g^{\nu\alpha}\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} \tag{5}$$

This, together with the equations

$$\begin{array}{rcl} \Gamma^{\lambda} = g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} & = & \frac{1}{2}g^{\mu\nu}g^{\lambda\kappa}\left\{ \frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} + \frac{\partial g_{\kappa\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\kappa}} \right\} \\ g^{\lambda\kappa}\frac{\partial g_{\kappa\mu}}{\partial x^{\nu}} & = & -g_{\kappa\mu}\frac{\partial g^{\lambda\kappa}}{\partial x^{\nu}} \end{array} \tag{6}$$

Allows us to rewrite the harmonic coordinate condition (7) as:

$$\Gamma^{\nu} = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\kappa}}(\sqrt{g}g^{\lambda\kappa}) = 0 \Leftrightarrow \frac{\partial}{\partial x^{\kappa}}(\sqrt{g}g^{\lambda\kappa}) = 0 \tag{7}$$

The reader might wonder why this method of specifying coordinates is called *harmonic.* A function ϕ is said to be *harmonic* if $\widehat{\nabla}\phi = 0$ where:

$$\nabla_{\kappa}\nabla^{\kappa}\phi \equiv \nabla_{\kappa}(g^{\lambda\kappa}\nabla_{\lambda}\phi) = g^{\lambda\kappa}\frac{\partial^2\phi}{\partial x^{\lambda}\partial x^{\kappa}} - \Gamma^{\lambda}\frac{\partial\phi}{\partial x^{\lambda}} \tag{8}$$

Therefore, if $\Gamma^{\lambda} = 0$ then the coordinates can be regarded as harmonic functions ($\nabla_{\kappa}\nabla^{\kappa}x^{\mu} = 0$). This gives rise to the term *harmonic coordinates.*

2.3 Hyperbolic Formulation of the Einstein Field Equations

In this section I will rewrite the Einstein Field Equations in the form of a first-order hyperbolic system. That is to say, a system of form:

$$\frac{\partial A}{\partial t} + \frac{\partial B^i}{\partial x^i} = C \tag{9}$$

This reformulation is useful because there already exists a large body of tested methods for solving this type of equations. They come to us from the field of fluid dynamics. The first attempts at accomplishing this were made by L. Smarr in 1979.

[1] This is equation (3.4.7) in Wald p. 48

2.3.1 Harmonic Slicing

The derivation presented here was developed by Choquet-Bruhat and Ruggeri in 1983. It consists of making the following coordinate conditions:

$$\begin{aligned} \Gamma^0 &= -\nabla_\kappa \nabla^\kappa x^0 = 0 && (10) \\ g^{0i} &= 0 && (11) \end{aligned}$$

This is the harmonic condition introduced in the previous section, applied to the time coordinate. Please notice that *only* the time coordinate is harmonic. We can think of this process as taking slices along the harmonic time coordinate. Hence the term *harmonic slicing*.

With condition (11) we can simplify equation 1 to:

$$ds^2 = -\alpha^2 dt + g_{ij} dx^i dx^j \tag{12}$$

Now, using equation (7) we can rewrite the condition (10) as:

$$\frac{\partial}{\partial t}\left(\frac{\alpha}{\sqrt{g}}\right) = 0$$

This easily integrates to:

$$\alpha = C(x)\sqrt{g} \tag{13}$$

Where $C(x)$ is the integration constant.

2.3.2 Vacuum Solution

Even with harmonic slicing, solving the field equations is a formidable task. Therefore, I will apply a further simplification by restricting the remainder of the derivation to the vacuum equations. In other words:

$$\begin{aligned} G^{\mu\nu} &= 0 \\ \Leftrightarrow \quad R^{\mu\nu} &= \tfrac{1}{2}\left(\nabla_\kappa \nabla^\kappa g^{\mu\nu} + \tfrac{\partial \Gamma^\nu}{\partial x^\mu} + \tfrac{\partial \Gamma^\mu}{\partial x^\nu}\right) - \Gamma^{\mu\rho\sigma}\Gamma^\nu_{\rho\sigma} = 0 \end{aligned} \tag{14}$$

Notice that if $R^{\mu\nu} = 0$ then $R = R^\mu_\mu = 0$ and then $G^{\mu\nu} = 0$.

Our objective is to express these equations in the hyperbolic form of equation 9.

Using equations like (6) and the harmonic condition (7) one can derive the following formula:

$$
\begin{aligned}
\Gamma^{\nu} &= \tfrac{1}{2} g^{\lambda\kappa} g_{\alpha\beta} \tfrac{\partial g^{\alpha\beta}}{\partial x^{\kappa}} - \tfrac{\partial g^{\lambda\kappa}}{\partial x^{\kappa}} \\
\Gamma^{i} &= \tfrac{1}{2} g^{ij} g_{\alpha\beta} \tfrac{\partial g^{\alpha\beta}}{\partial x^{j}} + \tfrac{1}{2} g^{i0} g_{\alpha\beta} \tfrac{\partial g^{\alpha\beta}}{\partial x^{0}} - \tfrac{\partial g^{ij}}{\partial x^{j}} - \tfrac{\partial g^{i0}}{\partial x^{0}} \quad (since\, \Gamma^{0} = 0) \\
&= \tfrac{1}{2} g^{ij} g_{\alpha\beta} \tfrac{\partial g^{\alpha\beta}}{\partial x^{j}} - \tfrac{\partial g^{ij}}{\partial x^{j}} \quad (since\, g^{i0} = 0) \\
&= \tfrac{1}{2} g^{ij} g_{ab} \tfrac{\partial g^{ab}}{\partial x^{j}} + \tfrac{1}{2} g^{ij} g_{00} \tfrac{\partial g^{00}}{\partial x^{j}} - \tfrac{\partial g^{ij}}{\partial x^{j}} \\
&= \tfrac{1}{2} g_{ab} g^{ij} \tfrac{\partial g^{ab}}{\partial x^{j}} - \tfrac{1}{2} g^{ij} \tfrac{1}{(\alpha^2)} \tfrac{\partial (\alpha^2)}{\partial x^{j}} - \tfrac{\partial g^{ij}}{\partial x^{j}} \\
&= \tfrac{1}{2} g_{ab} g^{ij} \tfrac{\partial g^{ab}}{\partial x^{j}} - g^{ij} \tfrac{1}{\alpha} \tfrac{\partial \alpha}{\partial x^{j}} - \tfrac{\partial g^{ij}}{\partial x^{j}} \\
&= \tfrac{1}{2} g_{ab} g^{ij} \tfrac{\partial g^{ab}}{\partial x^{j}} - g^{ij} \tfrac{\partial ln(\alpha)}{\partial x^{j}} - \tfrac{\partial g^{ij}}{\partial x^{j}}
\end{aligned}
$$

Now we define the quantities $D^{ab}_{c} \equiv \frac{\partial g^{ab}}{\partial x^{c}}$ and $L_j \equiv \frac{\partial ln(\alpha)}{\partial x^{j}}$. With this we obtain:

$$
\Gamma^{i} = \frac{1}{2} g_{jk} D^{ijk} - L^{i} - D^{ki}_{k} \tag{15}
$$

Finally we introduce the quantities $Q^{ij} \equiv \frac{\partial g^{ij}}{\partial x^{0}}$and $Q = Q^{k}_{k}$.

After a very long and laborious set of computations[2], the vacuum equation (14) into:

$$
\begin{aligned}
\frac{\partial D^{ij}_{k}}{\partial x^{0}} &= \frac{\partial Q^{ij}}{\partial x^{k}} \\
\frac{\partial \Gamma^{i}}{\partial x^{0}} &= Q L^{i} - 2 Q^{i}_{j} L^{j} + \Gamma^{i}_{jk} Q^{jk}
\end{aligned} \tag{16}
$$

This is a hyperbolic formulation of the field equations. Notice that these equations resemble equation (9) from the beginning of this chapter.

2.3.3 Closing Remarks

The reformulation of the field equations into a hyperbolic system is of great importance for numerical relativity. This process is laborious, but it is important because it makes the field equations *solvable* with known techniques. Once the field equations are in the form (16), we can use the methods already developed for fluid dynamics.

As a final remark, I will mention that in recent years (1995) it has been possible to write a hyperbolic form of the field equations without the need for harmonic slicing. This permits us to apply different slicing methods suitable for particular problems and still use familiar numerical methods.

[2] This result is not difficult to obtain, but it is very laborious. A somewhat more complete description is available from (Thomzen).

3 A Simple Example

The hyperbolic equations 16 look rather daunting. Most of the terms in them are unfamiliar. In an effort to improve the situation I will now compute the main terms of this equation for the simplest non-trivial situation. The metric that I will use is a slight variation on the Schwarzschild solution:

$$ds^2 = -e^{2m}dt^2 + e^{2n}dr^2 + r^2 d\phi^2 + r^2 sin^2\phi d\theta^2 \tag{17}$$

Where m and n are functions of r and t (the traditional formulation has them as functions of r only). I will not actually get as far as the hyperbolic formula. That would belong in a discussion of actual numerical techniques. I will only derive the unfamiliar terms L_i, Γ^i, Q^{ij}, and D_k^{ij}.

If we compare equation 17 with our generic metric $ds^2 = -\alpha^2 dt^2 + g_{ij}dx^i dx^j$, we see that $\alpha = e^m$. Therefore, we obtain $L_i \equiv \frac{dln(\alpha)}{dx}$

$$L_i = \frac{dln(\alpha)}{dx^i} = \frac{dm}{dx^i}$$

The nonzero terms are:

$$\begin{aligned} L_t &= \frac{dm}{dt} \\ L_r &= \frac{dm}{dr} \end{aligned}$$

The Christoffel symbols $\Gamma^\mu_{\nu\lambda}$ can be calculated, and from those, the terms $\Gamma^\lambda \equiv g^{\lambda\kappa}\Gamma^\mu_{\kappa\mu}$ can be obtained:

$$\begin{aligned} \Gamma^r &= e^{-2m}\left(\tfrac{2}{r} + \tfrac{\partial}{\partial r}(m+n)\right) \\ \Gamma^\theta &= 0 \\ \Gamma^\phi &= r^{-2}cot\phi \end{aligned}$$

Recall that the harmonic condition says $\Gamma^0 = \Gamma^t = 0$. Next we use $Q^{ij} \equiv \frac{\partial g^{ij}}{\partial x^0}$ and $Q \equiv Q^i_i$ to compute

$$Q^{rr} = 2e^{2n}\frac{\partial n}{\partial t} = Q$$

Recall that the Latin indices only run through the spacial dimensions. Thus, Q^{rr} is the only non-vanishing term. Finally we can compute $D^{ab}_c \equiv \frac{\partial g^{ab}}{\partial x^c}$. The non-vanishing terms are:

$$
\begin{aligned}
D_r^{rr} &= 2e^{2n}\frac{\partial n}{\partial r} \\
D_r^{\theta\theta} &= 2r sin^2\phi \\
D_r^{\phi\phi} &= 2r \\
D_\phi^{\theta\theta} &= 2r^2 sin\phi cos\phi
\end{aligned}
$$

4 The Current State of Research

4.1 Introduction

A great deal of work has gone into developing the methods of Numerical Relativity. The Einstein Field Equations form a set of 10 coupled, nonlinear, hyperbolic-elliptic partial differential equations that contain any thousands of terms. Even with the aid of modern computers, solving them poses a great challenge.

A realistic 3D simulation based on the full Einstein equations is an enormous task: A single simulation of coalescing neutron stars or black holes would require more than a teraflop[3] per second for reasonable performance, and a terabyte of memory. Modeling a more complicated phenomenon, such as the collision of two black-holes is simply beyond our current abilities. That is unfortunate because it is only such collisions that produce gravity waves strong enough that we might hope to detect in the foreseeable future.

4.2 The Cactus Code

In order to respond to this challenge, may institutions in the scientific community have joined forces in order to, concurrently, develop a solid set of computer codes that can eventually be used to solve the field equations. The **Cactus Code** is an open source (GNU General Public License) problem solving environment designed for scientists and engineers. It is developed by about fifty institutions spread throughout the globe[4]. Thanks to the Open Source methodology, the cactus code provides the power needed to solve the problems of Numerical Relativity, yet it is flexible enough to be useful for many other fields.

The name "Cactus" comes from its fundamental design. There is a small core kernel (the "flesh") and a large number of pluggable modules (the "thorns"). The flesh controls how the thorns work together, coordinating data-flow between

[3]flop = "floating point operation". The latest PCs from Intel and AMD perform about 10 giga-flops per second.

[4]See listing `http://sdcd.gsfc.nasa.gov/ESS/esmf_tasc/Files/Cactus_b.html`for a listing.

them and deciding when any routine should be executed. This modular design makes the platform both powerful and efficient.

Another appealing feature of Cactus is its portability. Cactus runs on most architectures. This allows the programmer to write and test code on his Linux or Mac OS X laptop and then bring it to a Cray T3E without alterations. The Cactus code will work fine on the single-CPU system (the laptop) and when brought to the Cray, it will take full advantage of that machine's parallel processing abilities.

As way of example, below are some snap shots of a simulation of the collision of two neutron stars. It was demonstrated in the Supercomputing'98 conference in Orlando. This simulation was made by two CRAY T3E's: one in Berlin and the other in San Diego.

From the NCSA website:

> Each of these Crays was computing on its own domain, such that the simulation started with one neutron star on the Berlin machine, the other one on the San Diego machine. So the collision was 'virtually' somewhere above the Atlantic ocean.

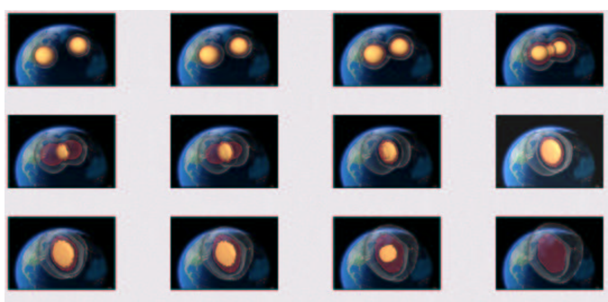

Figure 1: Collision of two Neutron Stars.
Credit: International Numerical Relativity Group, NCSA

This example illustrates how different agencies across the globe can collaborate to solve interesting problems. The Cactus code can be a significant catalyst for collaboration, by providing a common platform for development across different institutes.

The Cactus code goes much further than providing a common platform. Its open source nature allows for the mutual collaboration of all these agencies in developing the platform itself. Hence, it pulls together the efforts of a large number of interested organizations. This potential for collaboration is the basis of the appeal of the Cactus Code.

Numerical Relativity is a field under heavy development. It is not yet clear which techniques will be best suited for solving the problems that it wishes to address. This points to another appeal of the Cactus code. Its modular nature allows developers to try new methods and ideas very easily. As a result, it constitutes a significant catalyst in the development of such techniques.

4.3 Concluding Remarks

The Cactus code pulls together the efforts of individuals and institutions from all over the world and from a wide range of professional backgrounds. It offers them the flexibility to try new ideas easily, and the ability to collaborate. Without doubt, this joint effort will play an important role in the eventual solution of the Einstein Field Equations. With that, the Cactus project will help us learn about the nature of our universe, as expressed in the field equations. Furthermore, it will provide us with useful predictions for testing the theory of General Relativity in the realm of strong gravitational fields.

References

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