Hawking Radiation

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Caution Do not Read without Protective Eyeware

1 Klein-Gordon

The notion of Hawking Radiation was originally presented in a 1975 paper, published in the journal Communications in Mathematical Physics [2]. The essence of it is that particles are created by black holes, and that the distribution , corresponds to the distribution of energy as a function of frequency of a black body. The physical phenomenon is difficult to understand, but the the idea can be clearly expressed mathematically. This paper presents a step by step mathematical derivation of the Hawking Radiation equation. It is an elaboration of the paper by LH Ford [1].

First, we have to lay the foundation. We are considering an asymptotically flat space. This means that there is an interaction region, outside of which, the metric approximates Minkowski space. Specifically, we are considering the asymptotically flat Schwarzschild metric:

$$g = -(1 - 2M/r)\, dt^2 + dr^2/(1 - 2M/r) + r^2\, (d\theta^2 + sin^2\theta\, d\phi^2) \tag{1}$$

Note that the spatial part is written in spherical coordinates. One can verify that it is indeed asymptotically flat, by noting that, as r approaches infinity, the metric looks like Minkowski space;

$$g = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2}$$

where the differences stem from the fact that the Schwarzschild metric is described in spherical coordinates.

Calculating the g- matrix is straightforward: g is a (0,4) tensor (the four covectors $\omega^0 \dots \omega^3$ are $\{dt, dr, d\theta, d\phi\}$ The metric g is g_{ij} is $g_{ij}\omega^i \otimes\ \omega^j$

The off diagonal elements are zero and the diagonal elements are these:

$$g_{00} = -(1 - 2M/r),\ \ g_{11} = 1/(1 - 2M/r),\ \ g_{22} = r^2,\ \ g_{33} = r^2 sin^2\theta \tag{3}$$

The second ingredient we need, is the Klein Gordon Equation and some accompanying definitions. The Klein Gordon equation defines a scalar field. That is, a function that at each point on the manifold, gives a number. The Klein Gordon equation is found by determining the the stationary points of the Action Integral. The Action Integral, for its part, comes from the Lagrangian density which has to be determined. Here is the Lagrangian for our model.

$$L = 1/2\, (\partial_\alpha \phi \partial^\alpha \phi - m^2\phi^2 - \xi R \phi^2\ ) \tag{4}$$

Of course, ∂^α is just $g^{\alpha,\beta}\partial_\beta$. Setting

$$\frac{dA}{d\psi} = 0 \tag{5}$$

as was done in class, we find that

$$\Box\phi + m^2\phi + \xi R\phi = 0 \tag{6}$$

R is the scalar curvature. ξ is the coupling constant which indicates the strength of the last term relative to the rest of the equation. The box is the D'Alembertian operator,

$$\nabla_\mu \nabla^\mu = \nabla_\mu g^{\mu,k} \nabla_k. \tag{7}$$

Finally, note that in class, what we called $V(\phi) = -m^2\phi^2 - \xi R\phi^2$. So this is the Klein Gordon Equation for a massive field (i.e. m nonzero). Happily, we are considering a massless field so m=0, which simplifies things a bit.

Now, any solution of this KG eqn. can be written as a linear combination of a set of basis solutions. Given two solutions to KG eqn., (f_1, f_2) we define the KG inner product

$$< f_1, f_2 >_{kg} = \int (f_2^* \partial_\mu f_1 - f_1 \partial_\mu f_2^* d\Sigma^\mu \tag{8}$$

where * denotes the complex conjugate. Note then, that a constant c will come out of the first slot as c, but out of the second, as $c*$. In the above definition, $d\Sigma$ is the volume element on a spacelike hypersurface and $d\Sigma^\mu = n^\mu d\Sigma$ where n^μ is a unit normal to the spacelike hypersurface. In case you forgot, a spacelike hypersurface is one in which every tangent vector is spacelike. This in turn means that $g(vector, vector) > 0$. Or conceptually, one cannot get from one point on the surface to another without travelling faster than the speed of light. See LH Ford [1] who shows that the inner product is independent of the choice of spacelike-hypersurface.

2 Quantum Mechanics

To properly understand Hawkings Radiation, we also need a basic understanding of quantum mechanics. In quantum, there are no particles, just probability distributions. The probability of a particle being in a given set B, at a time t, is given by the integral

$$\int_B |\psi(x, y, z, t)|^2 dxdydz \tag{9}$$

ψ here, is not a number. It is a function, called the state. It represents a particular probability distribution. As such, we require that the integral of ψ, over all of R^3 is 1. At a particular (x, t) $\psi(x, t)$ is a complex number. To understand what comes next, an analogy will be helpful. Imagine a coin. It might be fair it might not be. In fact, there are an infinite number of potential probability structures. Given a particular structure, that is a particular probability distribution function, all sorts of quantities can be calculated; expected values, variances, probabilities of particular outcomes etc. The ψ's are analogous to the possible probability structures for the coin. There are a whole bunch of them. Given a particular one, we can also make all sorts of calculations. However, because this is quantum theory, the calculations are not as straightforward. Here is what we do:

First, define the Hermitian Product of two wave functions ψ as follows.

$$< \psi_1, \psi_2 > = \int \psi_1(x, t)^* \psi_2(x, t) dx \tag{10}$$

In general, any observable random variable is characterized by taking a Hermitian operator T. That is, we associate a linear Hermitian operator with whatever it is we want to measure. A Hermitian operator is an operator T, such

that,

$$< Ta, b >=< a, Tb > . \tag{11}$$

In finite dimensions, the linear operator T is a matrix, and T* is its transpose and complex conjugate. As an example, consider the operator T which measures the position of the first coordinate of the particle on which ψ operates.

$$T_1(\psi)(x,t) = x^1\psi(x,t) \quad \text{where } x = (x^1, y^1, z^1) \tag{12}$$

It is the combination of the Hermitian operator and Hermitian product that allow us to make our calculations. In particular, given a state ψ, we can determine the expected value and variance of an operator T as follows

The expected value e is

$$\begin{aligned} e &= < \psi, T(\psi) > \qquad (13) \\ &= \int \psi(x,t)^* T(\psi)(x,t)dx \qquad (14) \end{aligned}$$

and the variance is given by

$$v = < \psi, (T - eI)^2(\psi) > \tag{15}$$

where

$$(T - eI)^2 = (T - eI) \circ (T - eI) \tag{16}$$

is the composition of the operator with itself. We shall illustrate this, by developing the earlier example, in which T_1 is

the Hermitian operator corresponding to measuring the position of the first coordinate of the particle. We found that

$$T_1(\psi)(x,t) = x^1\psi(x,t) \tag{17}$$

The expected value of the first coordinate is calculated as follows.

$$\begin{aligned} < \psi, T_1(\psi) > &= \int \psi(x,t)^* x^1(\psi(x,t))dx \qquad (18) \\ &= \int x^1 |\psi(x,t)|^2 dx \end{aligned}$$

because

$$|\psi(x,t)|^2 = \psi(x,t)\psi(x,t)^* \tag{19}$$

Now, it is important to realize that the only time you can get a consistent number out of this Hermitian operator is when ψ is an eigenstate. That is, when

$$T(\psi) = \lambda(\psi). \tag{20}$$

Why? Because when this condition is satisfied the expected value of ψ is a number with zero variance:

$$
\begin{aligned}
< \psi, T(\psi) > &= < \psi, \lambda\psi > && (21)\\
&= \lambda \int |\psi|^2 && (22)\\
&= \lambda && (23)
\end{aligned}
$$

and the variance, as promised is,

$$
\begin{aligned}
v &= < \psi, (T - \lambda I)^2(\psi) > && (24)\\
&= < \psi, 0 > && (25)\\
&= 0 && (26)
\end{aligned}
$$

So we can measure using the linear operator to get out a number. But, again, for this

to work, ψ must be an eigenstate of the operator. As it turns out, position operators do not have eigenvalues. This is sad because if they did, the variance of the position operator would be zero, meaning the location of a particle could be found with precision. But you can't have it all.

Before we proceed any further, some notational issues should be highlited. What mathematicians call $< \psi_1, T(\psi_2) >$, physicists write as $< \psi_1|T|\psi_2 >$. $< \psi_1$ is called the Bra-Vector and corresponds to covectors. $\psi_2 >$ is the ket vector and corresponds to vectors. Anyway, so far we have dealt with a single particle. Now, consider n particles. The probability of particle 1 with coordinates $x_1 = (x^1, y^1, z^1)$ in A, and particle 2 with coordinates $x_2 = (x^2, y^2, z^2)$ in B is given by

$$\int_A \int_B |\psi(x_1, x_2)|^2 dx^1 dy^1 dz^1 dx^2 dy^2 dz^2 \tag{27}$$

As in the one particle case, the ψ is called a state and represents a probability distribution. The thing is, that this n particle wave function ψ can be expressed as a tensor product. In other words, it can be formed by a linear combination of orthonormal one particle states.

$$\psi = c^{\mu_1 \dots \mu_n} \psi_{\mu_1} \otimes \psi_{\mu_2} \dots \otimes \psi_{\mu_n} \tag{28}$$

In this equation ψ_{μ_1} means $\psi_{\mu_1}(x_1, t)$ where x_1 is a three space coordinate vector. If there were only a finite number S of states potentially this tensor could have S^n coefficients but physicists find that this structure is too general. In fact, wave functions fall into one of two subspaces. They are either symmetric or antisymmetric functions.

A symmetric wave function is of the form

$$\psi(x_1, ..., x_n) = \psi(x_{\sigma(1)}, ..., x_{\sigma(n)}) \tag{29}$$

where σ is a permutation of the entries. So for example in dealing with two particles

$$\psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2) \tag{30}$$

This symmetry has very clear consequences. Ordinarily, a tensor b does not equal b tensor a. So ordinarily this linear combination of tensor products would require specific knowledge of each term. However when the functions are symmetric you wind up summing all the permutations, a tensor b and b tensor a. So the order is unimportant. Once you know you have to use all the permutations the only information you need is that a and b are present. Similarly, once we know the wave function is symmetric all we need to characterize it, are the number of particles in each state. Particles with symmetric wave functions are called Bosons, and it is with Bosons that we will be dealing in this paper.

3 Occupation Numbers, Creation and Annihilation Operators

We are dealing with symmetric wave functions. We saw this meant that to characterize a basis for the symmetric wave functions we need only know the number of particles in each 1 particle state. We can create a vector space containing precisely this information. Construct the Occupation Number Space by defining an orthonormal basis, each element of which, is characterized by a sequence of integers greater than or equal to 0.

$$(n_1, n_2, \ldots) \tag{31}$$

Where, for instance, n_2 represents the number of particles in the 2nd state. Each wave function has an infinite number of potential states and each state may contain more than one particle. Thus, with n particles, there are at most n nonzero states. Following this notation, $\sum_{i=1}^{\infty} n_i$ is the total number of particles present. Of course it is possible that no particles are present at all, in any states. This arrangement is called the vacuum state.

We need a little more background before we can competently discuss Hawking Radiation. Define the annihilation operator a as follows. a_m removes one particle from the mth state if there is one there. If it

is empty it returns the zero vector. a_m also introduces a scale factor of $\sqrt{n_m}$. That is, the square root of the number of particles in the mth state. e.g.

$$a_3(0, 0, 2) = \sqrt{2}(0, 0, 1) \tag{32}$$

Similarly, define the creation operator $a_m^\dagger$ as follows. $a_m^\dagger$ adds 1 particle to the mth state. It introduces a scale factor equal to $\sqrt{(}n_m + 1)$ e.g

$$a_m^\dagger(0, 0, 2) = (0, 0, 3)\sqrt{n_m + 1} = \sqrt{3}(0, 0, 3) \tag{33}$$

The reason that the scale factors have been introduced, is most clearly explained by an example.

$$(a_2^\dagger a_2)(1, 2, 2) = 2(1, 2, 2) \tag{34}$$

So the composition of the creation and annihilation operators on the same state pulls out the number of particles in that state. This number is an eigenvalue. So the composition operator measures the number of particles in the mth state. The scale factors are constructed to produce this effect.

4 It all comes together

Now we are going to start putting these concepts together. Recall, that we have this Klein Gordon equation describing action. The Klein Gordon equation is a partial differential equation. Its solutions are functions which give complex numbers at a particular point. Consider a solution f to the KG equation eqn(6). Using the Klein Gordon inner product, defined in equation (8), we say that f is a positive norm solution if the KG inner product $< f, f >_{kg} > 0$ Otherwise, it is negative.

If f_j is a complete set of positive solutions, (f_j^*) is a complete set of negative solutions. Together, they make up a complete set of solutions to the wave equation. They are a basis. An arbitrary solution ϕ can then be written

$$\phi = \sum_j a_j f_j + a_j^\dagger f_j^* \tag{35}$$

Think of f_j, f_j^* as vectors for the moment, and the a_j and $a_j^\dagger$ as constants. This is a familiar form from linear algebra. But here, because of the Quantum Physics, there are some modifications when we quantize the equation. $\phi(x,t)$ isn't a number but an operator. Its a field operator which we expand in terms of the operators a_j and $a_j^\dagger$ instead of numbers.

This expansion can be done uniquely only in Minkowski Space, or equivalently, in asymptotically flat space, but not in curved spacetime.

Fortunately for us, we are considering space time which is flat in both the past and the future so we can use this expansion. Now to develop this a little further, let f_j be the set of positive solutions in the past (inmodes), and F_j be the set of positive solutions in the future (outmodes). We can also expand ϕ in terms of these outmodes.

$$\phi = \sum_k b_k f_k + a_k^\dagger f_k^* \tag{36}$$

The b's here are the annihilation and creation operators in the future. Before proceeding further, we place the following restrictions to ensure the basis solutions are orthonormal.

$$
\begin{aligned}
< f_j, f_{j'} >_{kg} &= \delta_{j,j'} && (37)\\
< F_j, F_{j'} >_{kg} &= \delta_{j,j'} && (38)\\
< f_j^*, f_{j'}^* >_{kg} &= -\delta_{j,j'} && (39)\\
< F_j^*, F_{j'}^* >_{kg} &= -\delta_{j,j'} && (40)\\
< f_j, f_{j'}^* >_{kg} &= 0 && (41)\\
< F_j, F_{j'}^* >_{kg} &= 0 && (42)\\
& && (43)
\end{aligned}
$$

We have constructed two bases for ϕ. The first, in terms of the past and the second, in terms of the future modes. Since both sets are a basis for all solutions, we can also expand the in-modes in terms of the out-modes.

$$f_j = \sum_k (\alpha_{jk} F_k + \beta_{jk} F_k^*) \tag{44}$$

Again, the f's are functions on the manifold, which give a complex number when evaluated at a particular point. The alpha and beta are complex number constants. We can also expand the outmodes in terms of the inmodes.

$$F_k = \sum_j (c_{jk} f_j + d_{jk} f_j^*) \tag{45}$$

But the coefficients c and d can be expressed in terms of α and β.

$$
\begin{aligned}
c_{lk} &= < F_k, f_l >_{kg} && (46)\\
&= < F_k, \sum_p (\alpha_{lp} F_p + \beta_{lp} F_p^*) > && (47)\\
&= \sum_p \alpha_{lp}^* < F_k, F_p > + \beta_{lp}^* < F_k, F_p^* > && (48)\\
&= \sum_p \alpha_{lp}^* \delta_{kp} \quad \text{using the orthonormal properties} && (49)\\
&= \alpha_{lk}^* && (50)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
d_{lk} &= - < F_k, f_l^* > && (51)\\
&= - < F_k, \sum_p (\alpha_{lp}^* F_p^* + \beta_{lp}^* F_p) > && (52)\\
&= - \sum_p \beta_{lp} < F_k, F_p > && (53)\\
&= - \sum_p \beta_{lp} \delta_{kp} \quad \text{using the orthonormal properties} && (54)\\
&= -\beta_{lk} && (55)
\end{aligned}
$$

So what we have is this.

$$F_k = \sum_j (\alpha^*_{jk} f_j - \beta_{jk} f^*_j) \tag{56}$$

This leads to a very important result.

$$\begin{aligned} < F_k, F_{k'} > &= \delta_{k,k'} && (57) \\ &= \sum_j \alpha^*_{jk}\alpha_{jk'} - \beta_{jk}\beta^*_{jk'} && (58) \end{aligned}$$

When k=k'

$$< F_k, F_{k'} >= \sum_j |\alpha_{jk}|^2 - |\beta_{jk}|^2 = \delta_{kk} = 1 \tag{59}$$

Building upon this, we can derive some other important results.

$$\begin{aligned} < f_j, f_{j'} >_{kg} &= < \sum_k (\alpha_{jk} F_k + \beta_{jk} F^*_k), \sum_r (\alpha_{j'r} F_r + \beta_{j',r} F^*_r) >_{kg} && (60) \\ &= \sum_r \sum_k \alpha_{jk}\alpha^*_{j'r} < F_k, F_r > + \beta_{jk}\beta^*_{j'r} < F^*_k, F^*_{r'} > && (61) \\ &= \text{The cross terms are zero because the solutions are orthogonal} && (62) \\ &= \sum_r \sum_k \alpha_{jk}\alpha^*_{j'r}\delta_{kr} + \beta_{jk}\beta^*_{j'r}(-\delta_{kr}) && (63) \\ &= \sum_k \alpha_{jk}\alpha^*_{j'k} - \beta_{jk}\beta^*_{j'k} && (64) \\ &= \delta_{jj'} && (65) \end{aligned}$$

The cross terms omitted in (61) are zero because the solutions are orthogonal. Also, in (63) remember, constants come out of the second slot of the KG inner product as their complex conjugate.

Substituting 44 into 39 and following a similar argument we find

$$\sum_k \alpha_{jk}\alpha_{j'k} - \beta_{jk}\beta_{j'k} = 0 \tag{66}$$

The only difference is that we use f^*_j so the constants that come out as complex conjugates are complex conjugates to begin with, so there are no complex conjugates in the final equation.

So we have found three equations relating the alphas and betas.

We can also express one set of annihilation and creation operators in terms of the other set. First, as explained earlier, the field operator can be expressed in terms of either the past or the future solutions. Following, 35

$$\phi = \sum_j a_j f_j + a^\dagger_j f^*_j = \sum_j b_j F_j + b^\dagger_j F^*_j \tag{67}$$

Now we probably should have mentioned this earlier, but a quantum field assigns to each point of the manifold a linear operator defined on a vector space with an inner product on that space. We shall call the vector space W, and denote the inner product on the space with the subscript W. The annihilation and creation operators, as part of our quantized expression for ϕ are of course defined for this space W. So, take vectors $w_1, w_2 \in W$, where $a_j(w_2) \in W$

Consider a function on the manifold which takes p to $K(p) =< w_1, \phi(p)w_2 >$

$$< w_1, \phi(p)w_2 >_w =< w_1, a_j w_2 >_w f_j(p) + < w_1, a_j^\dagger w_2 >_w f_j^*(p) \tag{68}$$

Let

$$\begin{aligned} n_j &= < w_1, a_j w_2 >_w && (69) \\ n_j^\dagger &= < w_1, a_j \dagger w_2 >_w && (70) \\ N_j &= < w_1, b_j w_2 >_w && (71) \\ N_j^\dagger &= < w_1, b_j \dagger w_2 >_w && (72) \end{aligned}$$

So

$$\begin{aligned} K(p) &= \sum_j n_j f_j(p) + n_l^\dagger f_l(p) && (73) \\ K(p) &= \sum_l N_l F_l(p) + N_j^\dagger F_j^*(p) && (74) \end{aligned}$$

Clearly both K and f_j are solutions to the Klein Gordon equation. We can therefore make use of the KG inner product. To find an expression for the n_j's

$$\begin{aligned} n_j &= < K, f_j >_{kg} && (75) \\ &= < \sum_l N_l F_l + N_l^\dagger f_l^*, \sum_k (\alpha_{jk} F_k + \beta_{jk} F_k^*) >_{kg} && (76) \\ &= \sum_l \sum_k N_l \alpha_{jk}^* \delta_{lk} - N_l^\dagger \beta_{jk} \delta_{lk} && (77) \\ &= \sum_l N_l \alpha_{jl}^* - N_l^\dagger \beta_{jl}^* && (78) \\ &= \sum_l < w_1, b_l w_2 >_w \alpha_{jl}^* - < w_1, b_l^\dagger w_2 >_w \beta_{jl}^* && (79) \end{aligned}$$

we bring the summation inside the inner product and use the fact that (80)
it is linear in the second term. (81)

$$= < w_1, [\sum_l \alpha_{jl}^* b_l - \beta_{jl}^* b_l^\dagger] w_2 >_w \tag{82}$$

But

$$n_j =< w_1, a_j w_2 >_w \tag{83}$$

And, because w_1 and w_2 are arbitrary

$$a_j = \sum_l \alpha^*_{jl} b_l - \beta^*_{jl} b^\dagger_l \tag{84}$$

Similarly,

$$b_k = \sum_j \alpha_{j,k} a_j + \beta^*_{j,k} a^\dagger_j \tag{85}$$

OK. Here is the point of all this. Consider the future creation and annihilation operators b_k, and $\dagger b_k$, Suppose we are dealing with the vacuum state defined earlier. Using 85,

$$\begin{aligned} b_k|0)_{in} &= b_k(invacuum) \\ &= \textstyle\sum_j (\alpha_{j,k}\, a_j|0)_{in} + \beta^*_{j,k}\, a^\dagger_j)|0)_{in} \quad \text{Using 85} \end{aligned} \tag{86}$$

But the term involving alpha is zero because a is an annihilation operator and we are operating on the vacuum state so there are no particles to annihilate.

Now recall from before, that the creation operator composed with the annihilation vector pulls out the average number of particles in a particular state. Therefore, if we wish to determine the number of particles in the kth mode of the instate we proceed as follows.

$$\begin{aligned} < 0_{in}, b^\dagger_k b_k 0_{in} > &= < b_k 0_{in}, b_k 0_{in} > \\ &= < \textstyle\sum_j \beta^*_{j,k} a^\dagger_j 0_{in}, \sum_l \beta^*_{l,k} a^\dagger_l 0_{in} > \\ &= \textstyle\sum_j \beta^*_{j,k} \sum_l \beta_{l,k} < a^\dagger_j 0_{in} a^\dagger_l 0_{in} > \\ &= \textstyle\sum_j \beta^*_{j,k} \sum_l \beta_{l,k} < 0_{in}, a_j a^\dagger_l 0_{in} > \\ &= \textstyle\sum_j \beta^*_{j,k} \beta_{j,k} \\ &= \textstyle\sum_j |\beta_{jk}|^2 \end{aligned} \tag{87}$$

The last step is legitimate because the product of a number and its complex conjugate is the same as the square of its absolute value, by the definition of the absolute value of a complex number. So what have we done? We have found the average number of particles in the kth mode in terms of the betas. This is crucial because we could not determine that number using the regular method. The reason is this, The instate vacuum is not an eigenstate of the out number operator and as explained we cannot get any eigenvalues out of it. But wait! Didnt I say earlier that the $\psi's$ are eigenstates of the number operator and that is the whole point of the thing? Yes, but that was for the instates. What we are doing now is expressing the instate vacuum as a linear combination of outs each of which is an eigenstate of the out state number operator and which therefore has an explicit number of particles. However, the linear combination of them is no longer an eigenstate of the outstate number operator. The reason being that the eigenvalue of each term of the combination is potentially different, and thus it is impossible to find an eigenvalue for the linear combination of them.

5 Particle Creation

We have now come to the heart of the matter. As we have stated repeatedly, the f's are solutions to the KG equation. At a particular point they give a complex number. It is generally known that these solutions take the following form.

$$f_{\omega lm} = \frac{Y_{lm}(\theta,\phi)}{r\sqrt{4\pi}}(\frac{1}{\sqrt{\omega}})e^{-i\omega v} \tag{88}$$

$$F_{\omega lm} = \frac{Y_{lm}(\theta,\phi)}{r\sqrt{4\pi}}(\frac{1}{\sqrt{\omega}}e^{-i\omega g(v)} \tag{89}$$

The f's are a function of frequency -ω and direction -lm.

We still need to understand the terms v and u in the above expression. To this end, let us retreat from the mathematics for a moment to some physical explanation. A black hole is formed by a collapsing star. However to simplify the proof we follow Ford [1] and model the phenomenon of a collapsing shell. We assume no particles were present prior to the collapse so the quantum state ψ is the in-vacuum.

So here is whats happening. Rays of light enter the black hole bounce around and then exit. The entering rays are said to come from past null infinity I^-. They then exit into positive null infinity I^+. However, not all the rays make it out. Some are trapped by the black hole. See figure 1. The line r=0 exterior represents the region of no return. Light rays which hit that line do not bounce out. From the diagram, we see that rays $w > v_0$ will become trapped. We will consider only rays which do escape (ie $v < v_0$]. As regards these rays, we want to know the relationship between the incoming ray parameterized by v and that same ray when it exits parameterized by u.

To do this we say the following. The region outside the black hole is described by the Schwarzschild metric, but the area of the black hole is approximated by Minkowski space. A light ray v enters the black hole from the past null infinity, and is then called V. As this ray enters the middle of the black hole and starts its journey out, we parameterize it by U. When it finally exits into the future null infinity, it shall be parametrized by u. So we have four stages in the journey and three transition stages. Assuring continuity at these transition points will establish the light rays path. First let us review the metrics we will use.

Schwarzschild:

$$g = -(1-2M/r)(dt^2) + dr^2/(1-2M/r) + r^2(d\theta^2 + sin^2\theta\, d\phi^2) \tag{90}$$

Minkowski:

$$g = -(dT^2) + dr^2 + r^2(d\theta^2 + sin^2\theta\, d\phi^2) \tag{91}$$

Now take another look at figure 1. It is called a Penrose diagram. It shows the path of a lightray in and out of a black hole. Actually, it isn't a direct representation of R and t. The Penrose diagram is a one-to-one transformation on the variables R and t with the angular coordinates supressed. Notice that the limiting value for an incoming ray (ie the last value for which the ray escapes) is v_0. The radius at this limiting value is denoted R_1 and the time T_1. Now turn back to the Minkowski metric, and rememeber that the angular portion can be disregarded. The definition of a light ray is that g(light ray) is zero. Setting the left side equal to 0 we find this condition to be that dT^2=dr^2 that is, T+r=V or T-r=U. In other words the light rays travel in straight lines at 45 degree angles. In fact all that is needed to characterize their path, are the intercepts U and V. This is for Minkowski space, inside the collapsing shell. Outside, where the Schwarzschild metric reigns supreme, the equivalent condition is that $T + r^* = v$ or $T - r^* = u$. where r^* turns out to be

$$r^* = r + 2M\ ln\left(\frac{r - 2M}{2M}\right) \tag{92}$$

So now we can begin tracing the lightray's path. First the ray enters the shell.

We want to develop the relationship between the ray as it enters (v) and the ray

once it is inside (V). The light ray goes from a region described by the Schwarzschild metric into Minkowski space. The continuity condition here on the surface of the black hole is this.

$$-1 + \left(\frac{dR}{dT}\right)^2 = -(1 - 2M/R)\left(\frac{dt}{dT}\right)^2 + (1 - 2M/R)^{-1}\left(\frac{dR}{dT}\right)^2 \tag{93}$$

The thinking behind this is as follows. Parameterize a point on the shell as follows and evaluate the tangent vector using the Minkowski metric.

$$\begin{aligned}
\gamma(T) &= (T, R(T)) && (94)\\
\gamma' &= (1\partial/\partial(T) + R'\partial/\partial r) && (95)\\
g(\gamma', \gamma') &= (-dT \otimes dT + dr \otimes dr)(\gamma', \gamma') && (96)\\
&= -1 + (R')^2 && (97)
\end{aligned}$$

So we have the left side of 93. When we apply the Schwarzschild metric to the same vector we get the right side. This is the matching condition. This leads us to the relationship between v and V. As r approaches R_1, dR/dT is approximately constant. So when we solve for dt/dT we find it is a constant. Integrating,

$$\begin{aligned}
t &= qT + s && (98)\\
T &= at + d && (99)\\
&\quad \text{but we saw earlier that} && (100)\\
v &= t + r^* && (101)\\
T &= a(v - r^*) + d && (102)\\
&\quad \text{but at this point } r^* \sim r \sim R1 \text{ a constant} && (103)\\
T &= av + f && (104)\\
&\quad \text{but we also saw} V = T + r. && (105)\\
&\quad \text{We substitute for T.} && (106)\\
V &= av + f + r \text{ where } r \sim R_1 && (107)\\
V &= av + b. && (108)
\end{aligned}$$

Now we proceed to the next transition point. At the center of the star V becomes U. The matching condition here is easy because at the center r=0. So U=T-r=T+r=V.

Last we consider the ray as it exits the black hole. Again we consider the case near the limiting condition. That is where $t = T_0$. We use a Taylor expansion of the function R which is the collapsing radius.

$$R(T) = 2M + A(T_0 - T) \tag{109}$$

where A is some constant. Now insert this into (93) to find that

$$\left(\frac{dt}{dT}\right)^2 = \left(\frac{R-2M}{2M}\right)^{-2}\left(\frac{dR}{dT}\right)^2 \approx \left(\frac{2M}{T-T_0}\right)^2 \tag{110}$$

Integrating it emerges that

$$t \sim -2M\ ln\left(\frac{T_0 - T}{2B}\right) \tag{111}$$

Now see 92. As r approaches 2M the second term of the right side becomes huge relative to the first term so we can disregard the r term. We can also substitute for $r - 2M$ using 109:

$$r^* \sim 2M\ ln\left(\frac{r-2M}{2M}\right) \sim 2M\ ln\left(A\frac{T_0 - T}{2M}\right) \tag{112}$$

Now that we have expressions for t and r^* we can use them in our expression for the parameterization of the light ray once it has left the Black Hole U. We saw earlier that $u = t - r^*$.

$$u = t - r^* \sim -4M\ ln\left(\frac{T_0 - T}{F}\right) \tag{113}$$

The term F in the denominator is a constant made up of the terms M,B and A.

We also saw earlier that U=T-R(T). Using 109 for R(T) yields

$$U = T - R(T) \sim (1+A)T - 2M - AT_0 \tag{114}$$

We can manipulate this to isolate $T_0 - T$.

$$T_0 - T = \frac{T_0 - U - 2M}{1+A} \tag{115}$$

Substituting this into 113 it is clear that

$$\begin{aligned} u &= -4M\ ln\left(\frac{T_0 - U - 2M}{(1+A)F}\right) && (116)\\ &= -4M\ ln\left(\frac{T_0 - [av+b] - 2M}{(1+A)F}\right) && (117)\\ &= -4M\ ln\left(\frac{(T_0 - b - 2M)/a - v}{(1+A)F/a}\right) && (118)\\ & && (119) \end{aligned}$$

Now, since $u \to \infty$ as $v \to v_0$

$$v_0 = (T_0 - b - 2M)/a \tag{120}$$

So we can conclude that

$$g(v) = -4M\ ln\left(\frac{v_0 - v}{C}\right) \tag{121}$$

M is the parameter for the mass of the star and v_0 is the limiting value of v for rays which pass through the black hole as has been explained. So using this expression for g(v) in 89 one gets

$$F_{\omega lm} = \begin{cases} \frac{Y_{lm}(\theta,\phi)}{r\sqrt{4\pi}}\frac{1}{\sqrt{\omega}}e^{4Mi\omega ln(\frac{v_0-v}{C})} & \text{if } v < v_0 \\ 0 & \text{if } v > v_0 \end{cases} \tag{122}$$

Again F takes on the value zero when $v > v_0$ because the ray does not escape the black hole. It has no representation in the future. It has no future! Using equation 56 as a guide we can

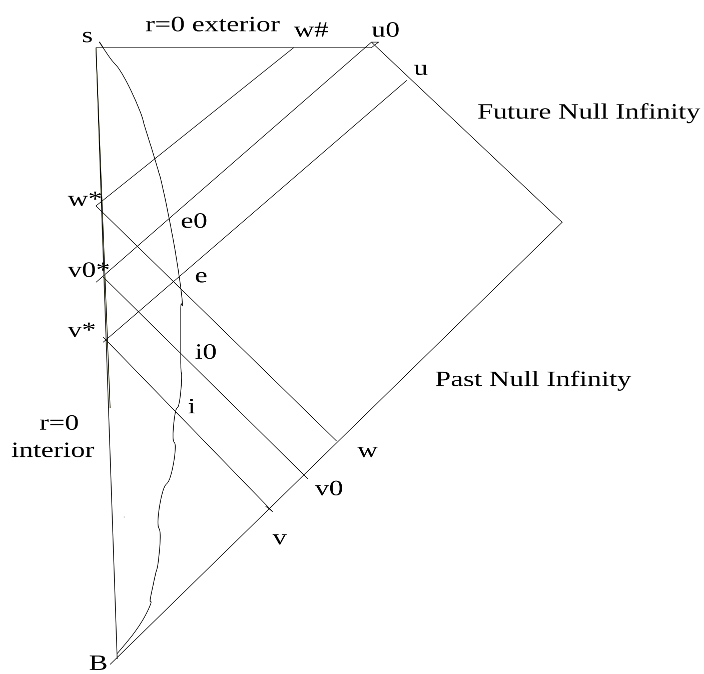

Figure 1: Penrose diagram

rewrite F_k as follows.

$$F_{\omega lm} = \int_0^\infty d\omega' \, (\alpha^*_{\omega' lm\, \omega lm} f_{\omega' lm} - \beta_{\omega' lm\, \omega lm} f^*_{\omega' lm}) \tag{123}$$

$$\tag{124}$$

We have changed the sum to an integral. The subscript k is now wlm while j is $w'lm$. We also have an expression for f from eqn122 which we can use here. 124then becomes,

$$F_{\omega lm} = \int_0^\infty d\omega' (\alpha^*_{\omega' lm\, \omega lm} \frac{Y_{lm}(\theta,\phi)}{r\sqrt{4\pi}} (\frac{1}{\sqrt{\omega'}}) e^{-i\omega' v} - \beta_{\omega' lm\, \omega lm} \frac{Y_{lm}(\theta,\phi)^*}{r\sqrt{4\pi}} (\frac{1}{\sqrt{\omega'}}) e^{i\omega' v} \tag{125}$$

Now two things should be apparent. One, equation 125 is in the form of a Fourier transform, where $F_{\omega lm}$ is $H(v)$. Two, since there are two terms in the integral we can break up our definition of $h(\omega')$.

$$h(\omega') = \begin{cases} \alpha^*_{\omega' lm\, \omega lm} \frac{Y_{lm}(\theta,\phi)}{r\sqrt{4\pi}} (\frac{1}{\sqrt{\omega'}}) & \text{if } \omega' \geq 0 \\ -\beta_{-\omega' lm\, \omega lm} \frac{Y_{lm}(\theta,\phi)^*}{r\sqrt{4\pi}} (\frac{1}{\sqrt{-\omega'}}) & \text{if } \omega' < 0 \end{cases} \tag{126}$$

We can thus rewrite eqn. 64 as follows.

$$H(v) = \int h(\omega') e^{-i\omega' v} d\omega' \tag{127}$$

The point of this little maneuver is that now we can now take advantage of the

inverse Fourier transform.

$$h(\omega') = \frac{1}{2\pi} \int H(v) e^{i\omega' v} dv \tag{128}$$

Specifically, consider the two cases of positive and negative ω'. If $\omega' \geq 0$

$$\alpha^*_{\omega' lm\, \omega lm} \frac{Y_{lm}(\theta,\phi)}{r\sqrt{4\pi}} \left(\frac{1}{\sqrt{\omega'}}\right) = \frac{1}{2\pi} \int_{-\infty}^{v_0} \frac{Y_{lm}(\theta,\phi)}{r\sqrt{4\pi}} (\frac{1}{\sqrt{\omega}} e^{-i\omega g(v)} e^{i\omega' v} dv \tag{129}$$

Cancelling and moving the ω' to the right we have

$$\alpha^*_{\omega' lm\, \omega lm} = \frac{1}{2\pi} \frac{\sqrt{\omega'}}{\sqrt{\omega}} \int_{-\infty}^{v_0} e^{-i\omega g(v)} e^{i\omega' v} dv \tag{130}$$

Similarly, we can isolate the beta's but it's a little more tricky. We will assume ω' is positive and take $-\omega'$. Watch out.

$$-\beta_{\omega' lm\, \omega lm} \frac{Y_{lm}(\theta,\phi)^*}{r\sqrt{4\pi}} \left(\frac{1}{\sqrt{\omega'}}\right) = \frac{1}{2\pi} \int_{-\infty}^{v_0} \frac{Y_{lm}(\theta,\phi)}{r\sqrt{4\pi}} (\frac{1}{\sqrt{\omega}} e^{-i\omega g(v)} e^{-i\omega' v} dv \tag{131}$$

$$\tag{132}$$

The trouble, of course, is that we have $\frac{Y_{lm}(\theta,\phi)^*}{r\sqrt{4\pi}}$ on the left, but $\frac{Y_{lm}(\theta,\phi)}{r\sqrt{4\pi}}$ on the right. Who says that these cancel? However, for reasons that will soon become apparent we actually are only interested in the absolute value of these quantities. The absolute value of a complex number $|a+bi| = \sqrt{a^2+b^2} = |a-bi|$. Thus the complex conjugate in the Y_{lm} term does not pose a problem.

Cancelling like terms, and moving the minus sign and the ω' to the right we have

$$-\beta_{\omega' lm\,\omega lm} = \frac{Y_{lm}}{Y^*_{lm}} \frac{1}{2\pi} \frac{\sqrt{\omega'}}{\sqrt{\omega}} \int_{-\infty}^{v_0} e^{-i\omega g(v)} e^{-i\omega' v} dv \tag{133}$$

OK, now here is what we are going to do. We want to find the relationship between $|\alpha_{\omega' lm\,\omega lm}|$ and $|\beta_{\omega' lm\,\omega lm}|$. Remember, that the absolute value of a number and the absolute value of its complex conjugate are the same. So, $|\alpha| = |\alpha^*|$. Also, recall that the absolute value of both e^{-ix} and e^{ix} is 1.

$$|\alpha^*_{\omega' lm\,\omega lm}| = \frac{1}{2\pi} \frac{\sqrt{\omega'}}{\sqrt{\omega}} \left| \int_{-\infty}^{v_0} e^{-i\omega g(v)} e^{i\omega' v} dv \right| \tag{134}$$

$$|\beta_{\omega' lm\,\omega lm}| = \frac{1}{2\pi} \frac{\sqrt{\omega'}}{\sqrt{\omega}} \left| \int_{-\infty}^{v_0} e^{-i\omega g(v)} e^{-i\omega' v} dv \right| \tag{135}$$

Notice that the terms of integration are negative infinity to v_0. The rationale is this. As explained earlier, at values of $v > v_0$ the light becomes trapped in the black hole and is lost. Now, we need to tinker with these equations a bit. First, the terms outside the integrals in 134 and 135cancel each other.Second, we make the following substitution:

$$v' = v_0 - v \tag{136}$$

$$dv' = -dv \tag{137}$$

This accomplishes three things. It alters the terms of integration. It allows us to absorb the negative sign by switching the upper and lower bounds of the integration. And it allows us to bring the terms involving v_0 in the exponent of e outside the integral. Thus we have

$$\int_{-\infty}^{v_0} e^{-i\omega g(v)} e^{i\omega' v} dv = \int_{-\infty}^{v_0} e^{i4M\omega\, ln(\frac{v_0 - v}{C})} e^{i\omega' v} dv \tag{138}$$

$$= e^{i\omega' v_0} \int_{0}^{\infty} e^{-i\omega' v'} e^{i4M\omega\, ln(\frac{v'}{C})} dv' \tag{139}$$

$$\int_{-\infty}^{v_0} e^{-i\omega g(v)} e^{-i\omega' v} dv = \int_{-\infty}^{v_0} e^{i4M\omega\, ln(\frac{v_0 - v}{C})} e^{-i\omega' v} dv \tag{140}$$

$$= e^{-i\omega' v_0} \int_{0}^{\infty} e^{i\omega' v'} e^{i4M\omega\, ln(\frac{v'}{C})} dv' \tag{141}$$

So find the relationship between the alphas and betas, it suffices to find the relationship between these two integrals. This can be done using closed contour integration. The result is this.

$$\int_{0}^{\infty} e^{-i\omega' v'} e^{i4M\omega\, ln(\frac{v'}{C})} dv' = -e^{4\pi M\omega} \int_{0}^{\infty} e^{i\omega' v'} e^{i4\pi M\omega\, ln(\frac{v'}{C})} dv' \tag{142}$$

The relationship then between the alphas and betas is this.

$$|\alpha_{\omega' lm\,\omega lm}| = e^{4\pi M\omega}|\beta_{\omega' lm\,\omega lm}| \tag{143}$$

But by 59, letting j=j', we conclude that:

$$1 = \sum_{\omega'} |\alpha_{\omega' lm\,\omega lm}|^2 - |\beta_{\omega' lm\,\omega lm}|^2 \tag{144}$$

$$= \sum_{\omega'} (e^{8\pi M\omega} - 1)|\beta_{\omega' lm\,\omega lm}|^2 \tag{145}$$

Now buckle your seatbelt. If we divide by the constant term we have solved for the betas in terms of known quantitites of mass and frequency. We have found the average number of particles created by a Black Hole.

$$\sum_{\omega'} |\beta_{\omega' lm\,\omega lm}|^2 = \frac{1}{e^{8\pi M\omega} - 1} \tag{146}$$

Let us relate this expression to the main term in the equation for black body radiation:

$$\frac{1}{e^{\frac{h\nu_{physical}}{kT}} - 1} \tag{147}$$

The $\frac{h\nu_{physical}}{kT}$ corresponds to our $8\pi M\omega$.

One final point. This last expression is fine mathematically. However, it has been simplified. The speed of light is 1 and all sorts of other constants have been adjusted to equal 1. If you want to actually calculate something some alterations are in order. The conversions are these

$$\omega = 2\pi\nu \tag{148}$$

$$\nu = \nu_{physical}/c \tag{149}$$

$$M = GM_{physical}/c^2 \tag{150}$$

so

$$8\pi M\omega = 8\pi\left(\frac{GM_{physical}}{c^2}\right)\frac{2\pi\nu_{physical}}{c} \tag{151}$$

The creation of these particles corresponds to the frequency distribution of a black body at temperature T satisfying

$$\frac{h\nu_{physical}}{kT} = 8\pi\left(\frac{GM_{physical}}{c^2}\right)\frac{2\pi\nu_{physical}}{c} \tag{152}$$

When we solve for T, the νs cancel. So we have

$$T_{physical} = hc^3/(16\pi^2 kGM_{physical}) \tag{153}$$

$$= \bar{h}c^3/(8\pi kGM_{physical}) \tag{154}$$

$$
\begin{aligned}
G &= 6.673 * 10^{-8}\, cm^3/gm\ sec^2 && (155)\\
k &= 1.3805 * 10^{-16}\, ergs/(degrees\ kelvin) && (156)\\
\bar{h} &= h/2\pi = 1.055 * 10^{-27}\, gm\ cm^2/sec && (157)\\
c &= 3 * 10^{10}\, cm/sec && (158)
\end{aligned}
$$

We have done the calculation for a black hole with the same mass as the sun. $1.989*10^{33}$gm:

$$T = 6.18 * 10^{-8}\, degrees\ kelvin \tag{159}$$

References

[1] Ford, L. Quantum Field Theory In Curved Spacetime. citeseer.nj.nec.com/ford97quantum.html.

[2] Hawking, S. W. Particle Creation by Black Holes . *Communications in Mathematical Physics 43* (1975), 199–220.