INVESTIGATING THE KERR BLACK HOLE USING MAPLE

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1 Introduction

1.1 Why Study the Kerr Black Hole

1.1.1 Overview of Black Holes

One consequence of the theory of General Relativity is that sufficiently massive, cold, spherical fluid bodies cannot exist in hydrostatic equilibrium and thus undergo complete gravitational collapse. The resulting geometry of the spacetime embodies a singularity. Intuitively, a spacetime singularity is where the curvature blows up. However, one encounters difficulties with such a definition. We shall return to this issue when we study the singularity of the Kerr black hole. The nature of a black hole is due to the singularity within, as is implied by the physical formulation of the Cosmic Sensor Conjecture (Penrose, 1979):

All physically reasonable spacetimes are globally hyperbolic, i.e., apart from a possible initial singularity (such as the "big bang" singularity) no singularity is ever visibly to any observer.

In other words, the spacetime curvatures is so large that not even light can escape. Hence the term black hole.

1.1.2 The Universality of the Kerr Black Hole

The Kerr black hole provides the only known stationary vacuum black hole solution to Einstein's equation. However, a remarkable result follows from theorems by Israel, Carter, Hawking and Robinson (1967-75), namely that the Kerr black hole is the only possible stationary vacuum black hole. Thus, consider a body which undergoes a gravitational collapse and under the Cosmic Sensor Conjecture forms a black hole. One expects that after a sufficient period of time the spacetime geometry would settle to a stationary final state and in addition that all matter present would be swallowed up leaving a vacuum, except possibly for electrostatic fields. If the Cosmic Sensor Conjecture is correct, then the end product must be a Kerr black hole determined only by the total mass and total angular momentum, irrespective of the original composition, shape and structure.

1.1.3 The Generality of the Kerr Black Hole

The Kerr black hole was proposed by Kerr in 1963. It describes a stationary axisymmetric vacuum solution to Einstein's equations. It introduces rotation into the static spherically symmetric solution proposed by Schwarzschild in 1916. The Kerr black hole was later generalized in 1965 by Newman et. al. to include charge, following a recipe by Reissner (1916) and Nordstrom (1918). This is believed to be the most general form of the stationary axisymmetric vacuum solution, and so, as suggested in the previous section, the charged Kerr black hole encompasses all the known stationary black hole solutions. The charged Kerr metric admits three parameters: e, a and m. The parameter e encodes the charge, the parameter a encodes the angular momentum per unit mass and the parameter m encodes the mass of the charged Kerr black hole. When $e = 0$, the spacetime metric reduces to the vacuum Kerr family of solutions. When $a = 0$, the spacetime metric reduces to the Reissner-Nordstrom solutions and when $a = e = 0$, the Schwarzschild metric is recovered. It appears that in any astrophysically reasonable situation $e \ll m$ and so we can neglect the effects on the electromagnetic fields on the geometry and focus on the Kerr family of solutions.

1.2 Why Use Maple

General Relativity by its nature is both symbolically involved and computationally intensive. Maple packages offer tools to assist in computations related to General Relativity. It allows the user to explore some of the known results of General Relativity, and also explore new domains by tensor definitions. For this paper, all Maple work was done with Maple V Release 4. Of course, later versions of Maple may be used as well. We briefly describe here three of the main packages for General Relativity computations with Maple. For a more detailed description, please refer to Appendix I.

1.2.1 The Tensor Package

The Tensor package is built into Maple V. It is an extensive package that contains commands for computing basic quantities such as the Christoffel symbols, the Riemann tensor and the curvatures, as well as some more advanced commands for handling Killing's equation and the Euler-Lagrange equations for the geodesic curves.

1.2.2 The Riemann Package

The Riemann package is an add-on package for Maple. In addition to commands for basic computations in General Relativity, it provides a good facility for the symbolic manipulation of tensors.

1.2.3 The GRTensorII Version 5 Package

The GRTensorII is an add-on package for Maple. In addition to commands for basic computations in General Relativity, it contains built in libraries for a number of known solutions to Einstein's equations, including the Schwarzschild solution and the Kerr solution. All of the computations for this paper where done with this package.

1.3 About this Paper

This paper set to investigate some of the main known results about the Kerr black hole with the aid of Maple. First, the basic properties of the Kerr metric are explored and then some of the characteristics black whole are studied. All computations for this paper are done in Boyer-Lindquist coordinates. These are the familiar t, r, θ and ϕ coordinates. However, the Kerr metric can also be explored in Maple via the Newman-Penrose (NP) tetrad formalism, especially using the GRTensorII package.

2 The Basic Properties of the Kerr Metric

In this section we study some of the basic properties of the Kerr metric in Boyer-Lindquist coordinates. We begin by clearing the workspace using 'restart', setting an interface parameter using 'interface' and invoking the GRTensorII Version 5 package using 'grtw()'.

```
> restart: interface(labelling=false): grtw();
```

$$GRTensorII\ Version\ 1.70\ (R4)$$

$$31\ May\ 1998$$

$$Developed\ by\ Peter\ Musgrave,\ Denis\ Pollney\ and\ Kayll\ Lake$$

$$Copyright\ 1994 - 1998\ by\ the\ authors.$$

$$Latest\ version\ available\ from:\ http://astro.queensu.ca/\tilde{\ }grtensor/$$

$$c:/GR/Grtii(4)/Metrics$$

Next, we load the built-in library for the Kerr metric in Boyer-Lindquist coordinates.

```
> qload(kerr);
```

$$Default\ spacetime = kerr$$

$$For\ the\ kerr\ spacetime:$$

$$Coordinates$$

$$\mathrm{x}(up)$$

$$x^{\ a} = [r,\, \theta,\, \phi,\, t]$$

$$Line\ \ element$$

$$ds^2 = \frac{(r^2 + a^2\cos(\theta)^2)\ \ d\,r^2}{r^2 - 2\,m\,r + a^2} + (r^2 + a^2\cos(\theta)^2)\ \ d\,\theta^2$$
$$+ \sin(\theta)^2\,(r^2 + a^2 + 2\,\frac{m\,r\,a^2\sin(\theta)^2}{r^2 + a^2\cos(\theta)^2})\ \ d\,\phi^2\ \ - 4\,\frac{m\,a\,r\sin(\theta)^2\ \ d\,\phi\ \ d\ \ t}{r^2 + a^2\cos(\theta)^2}$$
$$+ (-1 + 2\,\frac{m\,r}{r^2 + a^2\cos(\theta)^2})\ \ d\,t^2$$

$$Kerr\ \ metric\ \ in\ \ Boyer - Lindquist\ \ coordinates.$$

2.1 Confirming the Kerr Metric is a Vacuum Solution

We wish to show that the Kerr metric is indeed a vacuum solution of Einstein's equations. We begin by computing the Ricci tensor with both coordinate indices in covariant form. In order to do so, we invoke the 'grcalc' command with parameter 'R(dn,dn)'. The tensor name 'R' is reserved by GRTensorII for both the Ricci and the Riemann tensors, as is the case in the literature. If 'R' takes two parameters, then it is interpreted as the Ricci tensor, and if it takes four parameters, then it is interpreted as the Riemann tensor. The 'dn' indicates that we wish for the coordinate index to be represented in covariant form (with the contravariant form indicated by 'up').

```
> grcalc(R(dn,dn));
```

$$CPU\ \ Time\ = .234$$

Next, we simplify the expression. We use the command 'gralter', passing '_' as a parameter where the expression to be simplified is to be entered, indicating that the tensor just computed, namely 'R(dn,dn)', is to be simplified. The type of simplification - 'trig' indicates a trigonometric simplification.

```
> gralter(_,trig);

Component simplification of a GRTensorII object:

Applying routine `simplify[trig]` to object R(dn,dn)
```

$$CPU\ \ Time\ = .250$$

We can now display the Ricci tensor.

```
> grdisplay(_);
```

$$For\ the\ kerr\ spacetime:$$

Covariant Ricci

$$\mathrm{R}(dn,\ dn)$$

$$R_{\ a\ b} = \textit{All components are zero}$$

Indeed, this confirms that the Kerr metric is a vacuum solution to Einstein's equations. This agrees with the requirement stated in the introduction that in a black hole all matter present would be swallowed up leaving a vacuum except possibly for electrostatic fields.

2.2 The Case m = 0

We now wish to show that when the parameter m, interpreted as the mass of the Kerr black hole is zero, then we are naturally left with flat spacetime. This is accomplished by computing the Riemann tensor, substituting $m = 0$ and showing that the resulting tensor is equivalently zero.

First compute the Riemann tensor.

```
> grcalc(R(dn,dn,dn,dn));
```

$$\textit{CPU Time} = .265$$

Next, perform the substitution.

```
> grmap(_,subs,m=0,'x');
```

```
Applying routine subs to R(dn,dn,dn,dn)
```

Now simplify the expression.

```
> gralter(_,trig);
```

```
Component simplification of a GRTensorII object:

Applying routine 'simplify[trig]' to object R(dn,dn,dn,dn)
```

$$\textit{CPU Time} = .016$$

Display the result.

```
> grdisplay(_);
```

For the kerr spacetime :

Covariant Riemann

$$R(dn, dn, dn, dn) = \textit{All components are zero}$$

Indeed, we see that the Riemann tensor is equivalently zero. This confirms that the Kerr metric is flat in the limit $m = 0$.

2.3 The Symmetries of the Kerr Metric

GRTensorII has a utility for finding out the Killing vectors associated with a metric.

```
> KillingCoords():
```

```
Testing Killing coordinates for kerr
Created definition for coord1(dn)
Created a definition for    coord1(dn,cdn)
Created a definition for    coord1(up,cdn)
Created definition for coord2(dn)
Created a definition for    coord2(dn,cdn)
Created a definition for    coord2(up,cdn)
Created a definition for    coord3(up,cdn)
Created definition for coord3(dn)
Created a definition for    coord3(dn,cdn)
Created definition for coord4(dn)
Created a definition for    coord4(dn,cdn)
Created a definition for    coord4(up,cdn)
```

$$\mathit{CPU\ Time}\ = .657$$

$$\mathit{Killing\ Coordinate\ Test\ Results}$$

$$\mathit{Coordinate\ vector}\ = [r,\, \theta,\, \phi,\, t]$$

$$\mathrm{coord1}(up) = [1,\, 0,\, 0,\, 0],\ \ \mathit{not\ a\ Killing\ vector}.$$

$$\mathrm{coord2}(up) = [0,\, 1,\, 0,\, 0],\ \ \mathit{not\ a\ Killing\ vector}.$$

$$\mathrm{coord3}(up) = [0,\, 0,\, 1,\, 0],\ \ \mathit{a\ Killing\ vector}.$$

$$\mathrm{coord4}(up) = [0,\, 0,\, 0,\, 1],\ \ \mathit{a\ Killing\ vector}.$$

Since the coordinates t and ϕ do not appear in the metric, the coordinate vector fields ∂_t and ∂_ϕ are killing vector fields. The flow ∂_t consists of the coordinate translation that sends t to $t+\Delta t$ leaving the other coordinates fixed. These isometries express the time-invariance of the model. For ∂_ϕ, the flow consists of coordinate rotations that send ϕ to $\phi+\Delta\phi$. These isometries express the axial symmetry of the model.

2.4 Asymptotic Flatness

The Kerr black hole is said to be asymptotically flat. This can be seen crudely by taking the limit as $r \to \infty$ of the metric. The resulting metric is the Minkowski metric in spherical coordinates. In general, however, the notion of asymptotic flatness is non-trivial and a more rigorous justification is given by Ashtekar and Hansen (1978). Asymptotic flatness expresses the natural idea that far from the black hole its gravitational field is weak.

3 The Characteristics of the Kerr Black Hole

3.1 Singularity of the Kerr Black Hole

As mentioned in the introduction, determining the singularity of spacetime is by no means a trivial business. Following the intuitive notion that a singularity occurs at a place where the curvature blows up may lead to difficulties since, often a particular choice of a coordinate basis may cause the components of the Riemann tensor to behave badly even though a singularity may not be present at all. It turns out that a good way to determine the singularity of the Kerr black hole is to examine the Kretschmann scalar $R_{abcd}R^{abcd}$. In order to do that, we first need to perform a coordinate transformation $u = acos(\theta)$. GRTensorII has a built-in library for that, so we load it first.

```
> qload(newkerr);
```

$$Default\ spacetime = newkerr$$

$$For\ the\ newkerr\ spacetime:$$

$$Coordinates$$

$$\mathrm{x}(up)$$

$$x^{\ a} = [r,\, u,\, \phi,\, t]$$

$$Line\ element$$

$$ds^2 = \frac{(r^2+u^2)\ d\,r^2}{r^2 - 2\,m\,r + a^2} + \frac{(r^2+u^2)\ d\,u^2}{a^2-u^2} + \frac{(a^2-u^2)\,(r^2+a^2+2\,\frac{(a^2-u^2)\,m\,r}{r^2+u^2})\ \ d\,\phi^2}{a^2}$$
$$-\,4\,\frac{(a^2-u^2)\,m\,r\ \ d\,\phi\ \ d\ \,t}{a\,(r^2+u^2)} + (-1 + 2\,\frac{m\,r}{r^2+u^2})\ \ d\,t^2$$

$$The\ Kerr\ metric\ in\ Boyer-Lindquist\ type\ coordinates\ (u = acos(\theta)).$$

Now we calculate the Kretschmann scalar, referred to as *RiemSq*.

```
> grcalc(RiemSq);
```

```
Created definition for R(dn,dn,up,up)
```

$$CPU\ \ Time\ = .234$$

We simplify the expression.

```
> gralter(_,6,7);
```

```
Component simplification of a GRTensorII object:

Applying routine expand to object RiemSq
Applying routine factor to object RiemSq
```

$$CPU\ \ Time\ \ = .172$$

Now we can substitute back for u.

```
> grmap(_,subs,u=a*cos(theta),'x');
```

```
Applying routine subs to RiemSq
```

The resulting scalar is:

```
> grdisplay(_);
```

For the newkerr spacetime :

Full Contraction of Riemann

$$K \ = -48m^2\,(-r + a\cos(\theta))\,(r + a\cos(\theta))\,(r^2 - 4\,a\cos(\theta)\,r + a^2\cos(\theta)^2)$$
$$(r^2 + 4\,a\cos(\theta)\,r + a^2\cos(\theta)^2) \Big/ (r^2 + a^2\cos(\theta)^2)^6$$

The scalar blows up at $r^2 + a^2 cos(\theta)^2 = 0$. If the mass m is non-zero, this indicates a true singularity at $r = 0$ and $\theta = \pi/2$. Of course, this result only seems puzzling if we naively interpret the coordinate system to imply that our manifold structure is R^4. We get insight into the true nature of the singularity by studying the case $m = 0$ and $a \neq 0$. In this case, the Kerr metric is the Minkowski metric in spheroidal coordinates with a coordinate singularity on a ring of radius a in the plane $z = 0$. This suggests we define the Kerr metrics on a manifold in the neighborhood of the singularity having a topology $R^4 \setminus S^1 \times R$ (that is, take away a ring cross "time").

4 Concluding Remarks

4.1 Further Insight into the Kerr Black Hole

Current research into the Kerr black hole is plentiful. One of the more intriguing areas is that of energy extraction from the Kerr black hole. By definition, a black hole allows nothing to escape. And yet, by making a Kerr black hole absorbs a particle with negative total energy, positive energy may be extracted (Wald, 325-30).

Another fascinating area of research is that of black holes and thermodynamics. It seems there exists a striking parallelism between black hole dynamics and thermodynamics. Perhaps the most obvious similarity is between the second law of thermodynamics - the entropy of a thermally isolated system tends to increase, and the law of area increase of a black hole (Wald, 330-9).

4.2 General Relativity with Maple

I shall try to give here an overview of my impression of performing basic General Relativity computations with Maple.

4.2.1 The Strengths

- Lots of different packages to choose from
- Friendly interface for most packages
- Built-in libraries for some known solutions to Einstein's equations
- Flexibility in defining new tensors

4.2.2 The Weaknesses

- Would have been nice to have one well-coordinated package
- Limited flexibility in modifying the built-in libraries
- Limited accessability to internal data structures both for reading and writing data
- Limited help facility

4.2.3 Overall Assessment (based on the packages Tensor, Riemann and GRTensorII)

It seems like Maple is perfect for basic computations such as the Christoffel symbols, the Riemann tensor, the Ricci tensor and the curvatures, and it appears as a powerful tool at that. However, substantial effort need be put into getting results for some of the more fancy formulations such as computing the geodesic equations.

5 Appendix I - General Relativity with Maple

This appendix provides a listing of some of the Maple packages for General Relativity.

5.1 Tensor

- Description: "The Tensor package contains routines that deal with tensors, their operations, and their use in General Relativity both in the natural basis and in a moving frame. Some utilities to help manipulate tensors are also provided."

- Author: NA

- Source: built into Maple V Release 4.

5.2 Riemann

- Description: "Tools to manipulate tensor components, applications to General Relativity Theory, some symbolic manipulation tools."

- Author: Renalto Portugal

- Source: http://www.astro.queensu.ca/~portogal/Riemann.html

5.3 GRTensorII

- Description: "GRTensorII is a computer algebra package for performing calculations in the general area of differential geometry. Its purpose is the calculation of tensor components on curved spacetimes specified in terms of a metric or set of basis vectors. The package contains a library of standard definitions of a large number of commonly used curvature tensors, as well as the Newman-Penrose formalism. The standard object libraries are easily expandable by a facility for defining new tensors. Calculations can be carried out in spaces of arbitrary dimension, and in multiple spacetimes simultaneously. Though originally designed for use in the field of general relativity, GRTensorII is useful in many other fields."

- Author: NA

- Source: http://grtensor.phys.queensu.ca

5.4 Riegeom

- Description: Abstract tensor manipulation

- Author: Renalto Portugal

- Source: http://www.astro.queensu.ca/~portogal/Riegeom.html

5.5 NPTools

- Description: A package for tetrad formalism in General Relativity

- Authors: Sasha Cyganowski, John Carminati

- Source: http://www.cm.deakin.edu.au

5.6 Manifold

- Description: A package for integrating over manifolds

- Author: NA

- Source: http://sunsite.informatik.rwth-aachen.de/maple/frame02.htm