SPINORS AND THE DIRAC-EINSTEIN EQUATION

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ABSTRACT. In an attempt to discover an equation that satisfied Special Relativity as well as quantum mechanics, Dirac arrived at what is now known as the Dirac Equation. It will be discussed, semi-technically, in section (2). As a prelude we need to understand a little about the objects that the Dirac operator acts on, specifically spinors. We shall discuss these creatures in section (1). It was then concieved that upon merging the Dirac equations with the Einstein equation, we gain some insight into a theory of Quantum-Gravity. In section (3) we introduce Dirac's equation in curved space-time. After this, one is ready to see how the Einstein equations are combined with the Dirac Equations.

1. INTRODUCTION TO SPINORS

How difficult would it be to find an object that when rotated by 2π, looks exactly as when you started? Trivial ofcourse. Now, could you find an object that when rotated by 2π, has a different orientation then when you started? Objects that have the same orientation after a 2π rotation are known as spin-1 objects. Being overly simplistic, we can say that particles that do not tranform in this way are known as spinors. As a great example of such an object look up Feynman's "Waiter with a platter" trick on pg. 29 of [F,W]. As something that you can do yourself, try the following. Take a belt and loop one end around the top of a chair. Holding the free end rotate it by 4π. Your task is to uncoil it without rotating the belt or moving the chair. You can attempt the same thing with a 2π rotation but it will be in vain. A little more about that later. [The way you uncoil the belt is just by looping it over its free end.]

From the above we can begin to imagine what type of objects these spinors are. They can be viewed as objects that are tethered to some other structure. This forces an iterdependence on the way the object transforms and the way the "leash" transforms. Some mathematics may make this clearer.

I will be brief but attempt to motivate the rigorous ideas behind spinors. Here, we will be working on Minkowski's light-cone V where it will be realized as cross sections of the Riemann sphere. To begin recall that a vector U is null if

$$\|U\| = U \cdot U = U^i U^j g_{ij} = (U^0)^2 - (U^1)^2 - (U^2)^2 - (U^3)^2 = 0$$

We imagine U as starting from the origin of the cone and extending along the surface to some point ζ where $\zeta = \frac{\xi}{\eta}$ and $\xi,\eta \in \mathbb{C}$. Let the pair (ξ,η) be coordinates for some future-pointing null vector $\mathbf{K}$. Now, recall that the standard stereographic representation of the unit-sphere in (x, y, z)-space is given by:

$$x = \frac{\zeta+\bar{\zeta}}{\zeta\bar{\zeta}+1} \quad y = \frac{\zeta-\bar{\zeta}}{i(\zeta\bar{\zeta}+1)} \quad z = \frac{\zeta\bar{\zeta}-1}{\zeta\bar{\zeta}+1}.$$

These equations written in terms of ξ and η become:

$$x = \frac{\xi\bar{\eta} + \eta\bar{\xi}}{\xi\bar{\xi} + \eta\bar{\eta}} \tag{1}$$

$$y = \frac{\xi\bar{\eta} - \eta\bar{\xi}}{i(\xi\bar{\xi} + \eta\bar{\eta})} \tag{2}$$

$$z = \frac{\xi\bar{\xi} - \eta\bar{\eta}}{\xi\bar{\xi} + \eta\bar{\eta}}. \tag{3}$$

So in this way, we can define some point on the rim of the cone, i.e. at t=1. Let P be such a point with coordinates $(1, x, y, z)$.

Now consider the line $\vec{OP}$ where O is the origin. We can choose some other point R on $\vec{OP}$ with coordinates (T, X, Y, Z) where :

$$T = \xi\bar{\xi} + \eta\bar{\eta} \tag{4}$$

$$X = \xi\bar{\eta} + \eta\bar{\xi} \tag{5}$$

$$Y = -i(\xi\bar{\eta} - \eta\bar{\xi}) \tag{6}$$

$$Z = \xi\bar{\xi} - \eta\bar{\eta}. \tag{7}$$

This was obtained by just multiplying the coordinates of P by $\xi\bar{\xi}+\eta\bar{\eta}$. Let $K = \vec{OR}$ have the coordinates (T, X, Y, Z) as defined. We now define a point P' that also lies on the rim of the cone, i.e. it has coordinates $(1, x', y', z')$, such that it lies very near to P. We know that we can define a vector L defined by $L = \lim_{\varepsilon \to 0} \frac{\vec{PP'}}{\varepsilon}$. But a vector can be represented by a differential operator, i.e.

$$L = \lambda\frac{\partial}{\partial\zeta} + \bar{\lambda}\frac{\partial}{\partial\bar{\zeta}}. \tag{8}$$

Theorem1 λ is a numerical multiple of η^{-2}.
Proof:

We require λ to be some expression so that after applying a linear fractional transformation (see Ahlfors) to ζ we have $\tilde{\zeta}$ where

$$\tilde{\zeta} = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \tag{9}$$

and

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = 1. \tag{10}$$

So we want (*) $\tilde{\lambda}\frac{\partial}{\partial\tilde{\zeta}} + \bar{\tilde{\lambda}}\frac{\partial}{\partial\bar{\tilde{\zeta}}} = \lambda\frac{\partial}{\partial\zeta} + \bar{\lambda}\frac{\partial}{\partial\bar{\zeta}}$. (Note that these linear fractional transformations with the preceeding determinant property defines what is known as a spin transformation.)

Now, notice that

$$\frac{\partial\tilde{\zeta}}{\partial\zeta} = \left\{\frac{\alpha(\gamma\zeta + \delta) - \gamma(\alpha\zeta + \beta)}{(\gamma\zeta + \delta)^2}\right\} \tag{11}$$

$\Longrightarrow$

$$\frac{\partial}{\partial\zeta} = (\gamma\zeta + \delta)^{-}2\frac{\partial}{\partial\tilde{\zeta}} = \eta^2\tilde{\eta}^{-2}\frac{\partial}{\partial\tilde{\zeta}} \tag{12}$$

recalling that $\zeta = \frac{\xi}{\eta}$ and the determinant condition above.

Now substituting (5) into (*) we get

$$\tilde{\lambda}\frac{\partial}{\partial\tilde{\zeta}} + \bar{\tilde{\lambda}}\frac{\partial}{\partial\bar{\tilde{\zeta}}} = \lambda\eta^2\tilde{\eta}^{-2}\frac{\partial}{\partial\tilde{\zeta}} + \bar{\lambda}\frac{\partial}{\partial\bar{\tilde{\zeta}}} \tag{13}$$

$\Longrightarrow$

$$-\lambda\eta^2\frac{\partial}{\partial\tilde{\zeta}} = \left(\bar{\lambda}\frac{\partial}{\partial\bar{\tilde{\zeta}}} - \bar{\tilde{\lambda}}\frac{\partial}{\partial\bar{\tilde{\zeta}}} - \tilde{\lambda}\frac{\partial}{\partial\tilde{\zeta}}\right)\tilde{\eta}^2 \tag{14}$$

$\Longrightarrow$

$$\lambda\eta^2 = \tilde{\lambda}\tilde{\eta}^2 \tag{15}$$

And so λ is indeed a multiple of η^{-2}.♠

To make life easy we choose $\lambda = -\eta^{-2}$ which gives

$$\mathbf{L} = -\left(\eta^{-\mathbf{2}}\frac{\partial}{\partial\zeta} + \bar{\eta}^{-\mathbf{2}}\frac{\partial}{\partial\bar{\zeta}}\right). \tag{16}$$

This was all worth it as you will see in a few moments. We now have a vector **L** as well as a vector perpendicular to it, namely **K**. You can see this by noting that **L** is tangent to what we've been calling the rim and that its time component is zero. **K** on the other hand should be imagined as passing through P and with positive time components and spatial components perpendicular to those of **L**. (For details see section 1.4 of [P,R].)

So we are almost ready for the punchline. Consider the plane given by $m\mathbf{K} + n\mathbf{L}$ where $m, n \in \mathbb{R}$ and $n > 0$. We shall call this plane Π and refer to it and **K** collectively as a flag. It is natural then to call **K** the flag pole. Notice that Π is tangent to the cone and

Theorem(2) All directions in Π other than **K** are orthogonal to **K**.
Proof:

$$\mathbf{K}\cdot(m\mathbf{K}+n\mathbf{L}) = K^i(mK^j+nL^j)g_{ij} = mK^iK^jg_{ij}+nK^iL^jg_{ij} = 0 \tag{17}$$

recalling that **K** is a null vector.♠

So our flag now depends solely on **K** and **L**. Remembering that **L** depends on η^{-2} consider the transformation

$$\xi \longrightarrow \lambda\xi \qquad \text{and} \qquad \eta \longrightarrow \lambda\eta \tag{18}$$

where $\lambda = re^{i\theta}$.

We saw that to **K**, we can associate the coordinates (ξ, η). This means that under a transformation like (18) **K** will transform differntly than **L**. More specifically,

$$(18) \Longrightarrow \eta^{-2} \longrightarrow r^{-2}e^{-i2\theta}\eta^{-2} \tag{19}$$

This means that the flag will rotate by 2θ (about **K**) but **K** will only rotate by θ.

As an example consider taking θ from $0 \to \pi$. Clearly $(\xi, \eta) \to (-\xi, -\eta)$ but the flag plane Π will have rotated through 2π. And so Π will haveto rotate by 4π before the flag, i.e. **K** and Π, returns to its original state. This object we constructed is a 2-spinor and we may represent it as (ξ, η).

As a final word we should return to the belt trick. Why is it that it works when you twist the belt through 720 degrees but that you cannot undo the kink if it is formed with a 360 degree twist? Any rotation needs to identify an axis for which the rotation is occuring around. We can then imagine that to any rotation of θ radians one can identify a vector of length θ. So if we restrict ourselves to rotations $\theta \in [0, \pi]$, we can associate all rotations to a ball of radius π. We notice immediately though that whether you rotate about an axis **k** or -**k**, the effect is the same. Thus, we must associate antipodal points of this ball and we end up with an object whose topology is quite different from the ball. Namely, we have that any closed curve in this space can not be contracted to a point but any even multiple of it can be. So, one full twist cannot be contracted but two can be. This object is none other than SO(3). Therefore, to conclude, we call these complex vectors (ξ, η) on which SO(3) acts in this double valued way 2-component spinors.

2. Introduction to the Dirac Equation

Eventually there came a time when people were desiring to unite quantum mechanics with Special Relativity. These people wanted to preserve the wave mechanics of quantum theory, but wanted to incorporate Lorentzian covariance. So they wanted an equation whose solution would give a wave equation, i.e. one that describes the probability of finding a particle in a particular region of space, but at the same time they wanted the mathematics to remain faithful to Relativity. The first attempt to find such a creature has been dubbed the Klein-Gordon equation and is given by

$$\Box\psi = g^{jk}\partial_j\partial_k\psi = m^2\psi \tag{20}$$

for a particle of mass m. (Recall that $g_{jk} = diag(-1, 1, 1, 1)$) Dirac however, wanted an equation that was first order in time, like the Schrodinger equation. But in order to preserve Lorentzian covariance, this would mean that the whole equation would have to be first order. The attempt to find such an equation led to the Dirac Equation and this is the reason why it is sometimes referred to as the square-root of the Klein-Gordon Equation or the wave operator. Some mathematics will make this clearer.

We look for a differential operator

$$\not{\partial} = \gamma^j\partial_j \tag{21}$$

where the γ^j are some coefficients so that

$$\Box\psi = \not{\partial}(\not{\partial}\psi) \tag{22}$$

This is what we meant as the square root of the wave operator. We should notice something about ψ and the γ^j. Keeping (21) in mind we have Dirac's equation

$$\not{\partial}\psi = \gamma^j \partial\psi = m\psi \Longrightarrow \Box\psi = \not{\partial}(\not{\partial}\psi) = m^2\psi \tag{23}$$

So we require

$$\begin{aligned} \Box &= \not{\partial}\,\not{\partial} = (\gamma^j\partial_j)(\gamma^k\partial_k) \\ &= \gamma^j\gamma^k\partial_j\partial_k = \frac{1}{2}(\gamma^j\gamma^k + \gamma^k\gamma^j)\partial_j\partial_k \end{aligned} \tag{24}$$

which means, upon remembering (20), that

$$\gamma^j\gamma^k + \gamma^k\gamma^j = 2g^{jk}. \tag{25}$$

So taking $j = 1$ and $k = 2$ we have $\gamma^1\gamma^2 + \gamma^2\gamma^1 = 0$ which means that we have $\gamma^1\gamma^2 = -\gamma^2\gamma^1$. We see that the γ^i are not just ordinary numbers but turn out to be matrices. They are given by

$$\gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{26}$$

$$\gamma^j = \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & -\sigma^j \\ -\sigma^j & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix} \tag{27}$$

where $j = 1, 2, 3$ and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{28}$$

are known as the Pauli matrices. The ψ in (23) is a wave function made up of a column of 4 complex functions. It is known as a Dirac spinor and can be thought of as a two-spinor as described in the last section where each component is itself a 2-spinor, i.e $\psi = (\psi_a, \psi_b)$. To conclude this section we write the Dirac equation as

$$\gamma^0\frac{\partial\psi}{\partial t} + \gamma^1\frac{\partial\psi}{\partial x} + \gamma^2\frac{\partial\psi}{\partial y} + \gamma^2\frac{\partial\psi}{\partial z} + im\psi = 0 \tag{29}$$

For more details refer to [F].

3. The Dirac Equation in Curved Space Time

Recall from equation (25) that the Dirac matrices were related to the metric. Now, consider some observer in a curved space-time. She would look out her window and believe that, atleast locally, she was in some flat space-time. At any point about her, the Dirac matrices would be just like those in the previous section. This means that as functions of position, we have Dirac matrices $G^j(x) = \gamma^j$ for x a position on our space-time manifold. From equation (25) however we have

$$g^{jk} = \frac{1}{2}(G^j G^k + G^k G^j) \tag{30}$$

which means that any coordinate system that satisfies this, gives a local reference frame. This means that General Relativity can get along with the Dirac Operator G for curved space-time given by

$$G = iG^j(x)\frac{\partial}{\partial x^j} + B(x). \tag{31}$$

$B(x)$ is a matrix composed of spin connection coefficients. For details on this take a look at [Fin] A transformation of the form $\psi \longrightarrow U\psi U^{-1}$ for a unitary matrix U is known as a gauge transformation as well as $A \longrightarrow UAU^{-1} + U(\nabla U^{-1})$ where A is a matrix. In fact, a local change of basis is called a gauge transformation and connections are known as gauge fields. For mathematics to describe a real physical system we need the property that upon a local change in basis, the equations will remain invariant. So to summarize, one would find the spin derivative, i.e. the spin connection, by beginning with

$$D_j = \frac{\partial}{\partial x^j} - iC_j(x) \tag{32}$$

for the appropriate matrices C_j. Then you apply a gauge transformation and obtain some form for the C_j. More explicitly,

$$D_j \longrightarrow UD_jU^{-1} = \partial_j - iUC_jU^{-1} + U(\partial_j U^{-1}) \tag{33}$$

So we want $C_j(x)$ to satisfy

$$C_j \longrightarrow UC_jU^{-1} + iU(\partial_j U^{-1}) \tag{34}$$

which must be expressed in terms of the G^j and $B(x)$. After much work (again see [Fin]) it can be found that $B(x) = G^j(x)E_j(x)$ where

$$E_j = \frac{i}{2}\rho(\partial_j\rho) - \frac{i}{16}Tr(G^m\nabla_j G^n)G_mG_n + \frac{i}{8}Tr(\rho G_j\nabla_m G^m)\rho \tag{35}$$

and

$$\rho = \frac{i}{4!}\epsilon_{ijkl}G^iG^jG^kG^l. \tag{36}$$

For this paper we consider a spherically symmetric and static space-time. We can write the metric as

$$g_{ij} = diag(T^{-2}, -A^{-1}, -r^2, -r^2\sin^2\theta) \tag{37}$$

$$g^{ij} = diag(T^2, -A, -r^{-2}, -\frac{1}{r^2\sin^2\theta}) \tag{38}$$

and volume element as $\sqrt{|g|} = T^{-1}A^{-\frac{1}{2}}r^2|\sin\theta|$.

[Note that $A = A(r)$ and $T = T(r)$ are positive functions]

In order to find the Dirac operator in this metric, we need to know B(x) explicitly. We wish to transform the Dirac matrices so that equation (30) holds. Thanks to [F,S,Y1] we have

$$G^t = T\gamma^0 \tag{39}$$

$$G^r = \sqrt{A}(\gamma^1\cos\theta + \gamma^2\sin\theta\cos\phi + \gamma^3\sin\theta\sin\phi) \tag{40}$$

$$G^\theta = \frac{1}{r}(-\gamma^1\sin\theta + \gamma^2\cos\theta\cos\phi + \gamma^3\cos\theta\sin\phi) \tag{41}$$

$$G^\phi = \frac{1}{r\sin\theta}(-\gamma^2\sin\phi + \gamma^3\cos\phi). \tag{42}$$

Being armed with two weapons, namely Ricci's Lemma and a good computer we have

$$B = \frac{i}{2}\nabla_j G^j \tag{43}$$

[Note that Ricci's Lemma just states that $\nabla_\beta g_{\alpha\mu} = 0$]

It is now required to find what B looks like explicitly. We begin by noting that since our metric is static then $\partial_t\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{ab}\dot{g_{ab}} = 0$. So after simplifying we get

$$\frac{1}{\sqrt{|g|}}\partial_t(\sqrt{|g|}G^t) = 0$$

$$\frac{1}{\sqrt{|g|}}\partial_r(\sqrt{|g|}G^r) = (\frac{2}{r} - \frac{T'}{T})G^r$$

$$\frac{1}{\sqrt{|g|}}\partial_\theta(\sqrt{|g|}G^\theta) = \frac{1}{r\sin\theta}(-2\gamma^1\sin\theta\cos\theta + \gamma^2(\cos^2\theta - \sin^2\theta)\cos\phi + \gamma^3(\cos^2\theta - \sin^2\theta)\sin$$

$$\frac{1}{\sqrt{|g|}}\partial_\phi(\sqrt{|g|}G^\phi) = \frac{1}{r\sin\theta}(-\gamma^2\cos\phi - \gamma^3\sin\phi)$$

But this means that

(44) $$B = \frac{i}{r}(1 - A^{-\frac{1}{2}})G^r - \frac{i}{2}\frac{T'}{T}G^r$$

So finally, we have our Dirac operator

(45) $$G = iG^t\frac{\partial}{\partial t} + G^r(i\frac{\partial}{\partial r} + \frac{i}{r}(1 - A^{-\frac{1}{2}}) - \frac{i}{2}\frac{T'}{T}) + iG^\theta\frac{\partial}{\partial \theta} + iG^\phi\frac{\partial}{\partial \phi}.$$

4. The Dirac-Einstein Equations

We first rewrite the equations and simplify things greatly. It is then necessary to find the Energy-Momentum tensor and from there we write the Dirac-Einstein Equations.

To begin with, we introduce some notation.

(46) $$\sigma^r(\theta,\phi) = \sigma^1\cos\theta + \sigma^2\sin\theta\cos\phi + \sigma^3\sin\theta\sin\phi$$

(47) $$\sigma^\theta(\theta,\phi) = -\sigma^1\sin\theta + \sigma^2\cos\theta\cos\phi + \sigma^3\cos\theta\sin\phi$$

(48) $$\sigma^\phi(\theta,\phi) = \frac{1}{\sin\theta}(-\sigma^2\sin\phi + \sigma^3\cos\phi)$$

To simplify further we take as the ansatz a spinor of the form

(49) $$\Psi_a = e^{-i\omega t}\begin{pmatrix} u_1 e_a \\ \sigma^r u_2 e_a \end{pmatrix}$$

where $a = 1,2$ and $e_1 = (1,0)$, $e_2 = (0,1)$. The $u_a(r)$ are complex valued functions. This is all good because it allows us to write two independent Dirac equations

(50)
$$G\Psi_a = \left[\begin{pmatrix} 0 & \sigma^r \\ -\sigma^r & 0 \end{pmatrix}(i\sqrt{A}\partial_r + \frac{i}{r}(\sqrt{A}-1) - \frac{i}{2}\frac{T'\sqrt{a}}{T}) + \omega T\gamma^0 + \frac{2i}{r}\begin{pmatrix} 0 & \sigma^r \\ 0 & 0 \end{pmatrix}\right]\Psi_a$$

which is not obvious unless you note that

$$\sigma^\theta(\partial_\theta\sigma^r) = \sigma^\phi(\partial_\phi\sigma^r) = \mathbb{1}.$$

Now, let $\Phi_1 = rT^{-\frac{1}{2}}u_1$ and $\Phi_2 = -irT^{-\frac{1}{2}}u_2$. This allows to simplify further the radial dependence. Write

(51)
$$\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\omega T - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sqrt{A}\partial_r + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{1}{r} - m\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\Phi = 0.$$

Which can be rewritten as the ODE

$$\sqrt{A}\Phi' = \left[\omega T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{r}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\Phi \tag{52}$$

where this is arrived at by pulling out the second term in equation (51) and multiplying through by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

So equation (52) allows us to view the Dirac equation as a two component equation.

The Einstein Field Equations are given by

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}. \tag{53}$$

This means that in order to continue we are going to need the Energy-Momentum tensor T_{ab}. This tensor can be arrived at by looking at the variation of the Dirac action given by

$$S = \int \overline{\Psi}(G - m)\Psi\sqrt{|g|}d^4x \tag{54}$$

Since the action is real we just have to pay attention to the real portion of (54). We consider

$$\delta S = \int Re\overline{\Psi}(i(\delta G^j)\frac{\partial}{\partial x^j} + \delta B)\Psi\sqrt{|g|}d^4x \tag{55}$$

The variation of the matrix B can be shown to vanish. This leaves us with

$$\delta S = \int \frac{1}{2}\sum_{a=1}^{2} Re\overline{\Psi_a}(iG_j\frac{\partial}{\partial x^k})\Psi_a\delta g^{jk}\sqrt{|g|}d^4x \tag{56}$$

and so the Energy-Momentum tensor is the matrix

$$T_{jk} = \frac{1}{2}\sum_{a=1}^{2} Re\overline{\Psi_a}(iG_j\frac{\partial}{\partial x^k} + iG_k\frac{\partial}{\partial x^j})\Psi_a \tag{57}$$

Thanks to plenty of cancellations (please see [F,S,Y1]) we arrive to

$$\begin{aligned} T_{ij} = r^{-2}diag(&2\omega T^2|\Phi|^2, -2\omega T^2|\Phi|^2 + 4Tr^{-1}\Phi_1\Phi_2 + \\ &2mT(\Phi_1^2 - \Phi_2^2), -2Tr^{-1}\Phi_1\Phi_2, -2Tr^{-1}\Phi_1\Phi_2). \end{aligned} \tag{58}$$

Now, after some heavy calculating we found the G_{ab}. Specifically,

$$G_{00} = -\frac{1}{r^2} + \frac{A}{r^2} + \frac{A'}{r} \tag{59}$$

$$G_{11} = -\frac{1}{r^2} + \frac{A}{r^2} - \frac{2AT'}{rT} \tag{60}$$

$$G_{22} = G_{33} = \frac{A'}{2r} - \frac{AT'}{rT} - \frac{A'T'}{2T} + \frac{2AT'^2}{t^2} - \frac{AT''}{T}. \tag{61}$$

Therefore, using equation (53) and letting $\alpha = \Phi_1$ and $\beta = \Phi_2$ we have the Dirac equations, equation (52), expressed as

$$\sqrt{A}\alpha' = \frac{1}{r}\alpha - (\omega T + m)\beta \tag{62}$$

$$\sqrt{A}\beta' = (\omega T - m)\alpha - \frac{1}{r}\beta \tag{63}$$

and the Einstein field equations expressed as

$$rA' = 1 - A - 16\pi\omega T^2(\alpha^2 + \beta^2)$$

$$2rA\frac{T'}{T} = A - 1 - 16\pi\omega T^2(\alpha^2 + \beta^2) + 32\pi\frac{1}{r}T\alpha\beta + 16\pi m T(\alpha^2 - \beta^2).$$

If you have nothing to do for a long while you can attempt to show that equations (62) and (63) as well as the last field equations, are equivalent to

(64)

$$-16\pi T r^{-1}\Phi_1\Phi_2 = A\left[r^2\frac{T''}{T} + r^2\frac{A'T'}{2AT} - 2r^2\left(\frac{T'}{T}\right)^2 - r\frac{A'}{2A} + r\frac{T'}{T}\right].$$

This is exactly the Dirac-Einstein equation in a spherically symmetric space-time. The last four equations before this are much more useful if you are concerned with discovering the properties of solutions for (64). This is what Finster and company do in [F,S,Y1], where they go on to describe the stability of solutions. My work here was essentially an attempt to understand the beginning of [F,S,Y1] and [F,S,Y2] as well as several chapters of [P,R].

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