A REVIEW ON THE COSMOLOGICAL CONSTANT

S. H. LOUIS LEUNG

0. Introduction--Homogenous and Isotropic Universe

Intuitively speaking, a universe is homogeneous if every point in space "looks like" any other point while a universe is isotropic if at any given point, the universe looks "the same" in any direction. (Readers may refer to Wald for precise definitions.)

Suppose we have a homogeneous and isotropic universe, and let Σ_t be a space-like (3-dimensional) hypersurface in the universe. The (pseudo-Riemannian) spacetime metric g_{ab} induces a Riemannian metric $h_{ab}(t)$ on Σ_t.

Let ${}^{(3)}R_{abc}{}^{d}$ be the Riemann tensor constructed from h_{ab} on Σ_t. Raising the third index we may consider ${}^{(3)}R_{ab}{}^{cd}$ as a linear transformation L from the vector space W of 2-forms to itself. Since $R_{abcd} = R_{cdab}$, L is self-adjoint (with the positive-definite inner product defined on W by h_{ab}). If L has distinct eigenvalues, we would be able to pick out a preferred two-form and consequently, a preferred vector, at a point p on Σ_t, violating isotropy. Therefore all eigenvalues of L are the same. As a result we have (where K is a number and I is the identity operator),

$$L = KI\,, \tag{0.1}$$

that is

$${}^{(3)}R_{ab}{}^{cd} = Kd^{c}{}_{[a}d^{d}{}_{b]}\,. \tag{0.2}$$

Lowering the indices, we get

$${}^{(3)}R_{ab}{}^{cd} = Kh_{c[a}h_{b]d}\,. \tag{0.3}$$

Homogeneity implies that K must be a constant. A space where (0.3) is satisfied and K is constant is called a space of constant curvature. If K is positive, we have a 3-sphere. If K=0 we have the Euclidean 3-space and if K is negative we have a 3-dimensional hyperboloid.

1. Introducing the Cosmological Constant

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1.1. Motivation

Einstein's original field equation is

$$R_{mn} - \frac{1}{2}Rg_{mn} = 8pGT_{mn}\,. \tag{1.1}$$

On a very large scale the universe is spatially homogeneous and isotropic to a good approximation. Therefore the spatial metric takes the Robertson-Walker form

$$ds^2 = -dt^2 + a^2(t)R_0^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right], \tag{1.2}$$

where $d\Omega^2 = dq^2 + \sin^2 q\, df^2$ is the metric on a 2-sphere. The parameter k is +1, 0 or –1 depending on the sign of the curvature of the spatial section. The scale factor a

characterizes the relative size of the spatial section as a function of time. Here it is written in normalized form $a(t)=R(t)/R_0$, where R_0 is evaluated at the present time. (Here R is not the scalar curvature, but a scale factor measuring the size of the spatial section.) The stress-energy-momentum (SEM) tensor may be modeled as a perfect fluid, with energy density ρ and isotropic pressure p. We write $T_{\mu\nu}$ in the following way:

$$T_{\mu\nu} = (\rho + p)U_{\mu}U_{\nu} + pg_{\mu\nu}, \tag{1.3}$$

where U^{μ} is the fluid four-velocity. In the case of a Robertson-Walker solution, Einstein's equation reduces to the Friedmann equations

$$H^2 \equiv (\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2 R_0^2}, \tag{1.4}$$

where $H \equiv \ddot{a}/a$ is the Hubble parameter, and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{1.5}$$

To account for astronomical data known at the time, Einstein was interested in finding static solutions (where $\dot{a}=0$) to his equation. If the energy density is positive, from (1.4) we know that a static universe is possible if $k=+1$ (i.e. the spatial curvature is positive) with the parameters ρ and R_0 carefully adjusted. However (1.5) suggests that if the isotropic pressure is nonnegative (which is true for most forms of matter), $\ddot{a}$ does not vanish. As a result, Einstein proposed a modification to his equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.6}$$

where Λ is a free parameter. With this modification, the Friedmann equations become

$$H^2 \equiv (\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2 R_0^2}, \tag{1.7}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{1.8}$$

Now with these modifications Einstein's equation admits a static solution (with ρ, p and Λ nonnegative). This solution is called the "Einstein static universe."

The discovery of the exapansion of the universe by Hubble eliminated the empirical need for a static universe. However, (1.6) still seemed to be a legitimate modification to Einstein's field equation and a priori we do not have a reason to assume that $\Lambda=0$. Although the original motivation by Einstein disappeared, from particle theory a new motivation arises for a nonzero Λ term.

1.2. Vacuum Energy

If we have a single scalar field ϕ, the action S is given by

$$S = \int d^4x\sqrt{-g}[\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\,\partial_{\nu}\phi - V(\phi)], \tag{1.9}$$

where g is the determinant of $g^{\mu\nu}$, and the corresponding SEM tensor is

$$T_{mn} = \frac{1}{2}\partial_m f\ \partial_n f + \frac{1}{2}(g^{rs}\partial_r f\ \partial_s f)g_{mn} - V(f)g_{mn}\,. \qquad (1.10)$$

In the configuration with the lowest energy density (if one exists), there is no contribution from kinetic or gradient energy, which implies that $\partial_m f = 0$, and therefore $T_{mn} = -V(f_0)g_{mn}$, where f_0 minimizes V(f). We have no reason to suggest that V(f_0) =0, so we can write

$$T^{vac}_{mn} = -r_{vac}\, g_{mn}\,, \qquad (1.11)$$

where r_{vac} = V(f_0). Therefore the vacuum can be thought of as a perfect fluid as in (1.3), with p_{vac}=- r_{vac}.

If we let

$$T^{0}_{mn} = \frac{1}{2}\partial_m f\ \partial_n f + \frac{1}{2}(g^{rs}\partial_r f\ \partial_s f)g_{mn} \qquad (1.12)$$

(i.e. the SEM tensor assuming V(f_0)=0) and

$$\Lambda = \frac{r_{vac}}{8pG} \equiv \frac{r_{\Lambda}}{8pG}\,, \qquad (1.13)$$

then we can write

$$R_{mn} - \frac{1}{2}Rg_{mn} + \Lambda g_{mn} = 8pGT^{0}_{mn}\,. \qquad (1.14)$$

It is therefore easy to see that the effect of a nonzero energy density of the vacuum is equivalent to assuming zero energy density of the vaccuum and a nonzero cosmological constant.

It is not necessary to introduce scalar fields in order to obtain a nonzero vacuum energy. The action for general relativity with a "bare" cosmological constant Λ_0 is given by

$$S = \frac{1}{16pG}\int d^4x\sqrt{-g}\,(R - 2\Lambda_0) \qquad (1.15)$$

where R is the Ricci scalar. Extremizing this equation would yield (1.6), with $\Lambda_0 = \Lambda$. Therefore the cosmological constant can be considered as a constant term in the Lagrange density. In fact (1.15) is the most general covariant action we can construct using the metric and its first and second derivatives, and thus a natural starting point for our theory of relativity.

As a result, classically the effective cosmological constant is the sum of a bare term Λ_0 and the minimum energy V(f_0), which may vary as the universe evolves.

2. Cosmology

From the Friedmann equation (1.4) we know that for any Hubble parameter H, there is a critical value of the energy density such that the spatial geometry is flat (i.e. k=0):

$$r_{crit} = \frac{3H^2}{8pG}\,. \qquad (2.1)$$

We often write the energy density in terms of this critical value:

$$\Omega = \frac{r}{r_{crit}} . \tag{2.2}$$

The energy density includes contributions from various components r_i. It is often the case that individual components *i* satisfy the following equation:

$$p_i = w_i r_i , \tag{2.3}$$

where p_i's are components of the isotropic pressure, and w_i's are constants. Therefore by the energy-momentum conservation equation $\nabla_m T^{mn} = 0$, we get:

$$r_i \propto a^{-n_i} , \tag{2.4}$$

where a is the scale factor we mentioned above and n_i is given by:

$$n_i = 3(1+w_i). \tag{2.5}$$

Therefore we can define Ω_i by:

$$\Omega_i = \frac{r_i}{r_{crit}} . \tag{2.6}$$

Examples of these components include massive particles (with negligible relative velocities), radiation (this includes all relativistic particles and is therefore not restricted to photons), and vacuum energy. The energy density of massive particles is given by their number density multiplied by their rest mass, so we have $r_M \propto a^{-3}$. The energy density of radiation is given by the number density multiplied by the particle energy, which is proportional to a^{-1} (redshift due to the expansion of the universe). Therefore, we have $r_R \propto a^{-4}$. Vacuum energy does not change as the universe expands, so $r_\Lambda \propto a^0$. Looking back at equation (1.4), for some purposes it is useful to pretend that the $-ka^{-2}R_0^{-2}$ term represents an "effective energy density" due to curvature. If we define $r_k \equiv -(3k/8pGR_0^2)a^{-2}$ and divide (1.4) by H^2, we get

$$\Omega_k = 1 - \Omega \tag{2.7}$$

We have good reasons to believe that the energy density due to radiation is much less than that due to matter. Photons contribute $\Omega_g \approx 5\times10^{-5}$ to the energy density, mainly due to the 2.73K cosmic microwave background. According to our current knowledge of the neutrinos, their contribution is approximately the same amount. As a result, we can parametrize our universe only by Ω_M and Ω_Λ (therefore $\Omega_k = 1 - \Omega_M - \Omega_\Lambda$).

A measure of the evolution of the expansion rate is the deceleration parameter

$$\begin{aligned} q &\equiv -\frac{\ddot{a}a}{\dot{a}^2} \\ &= \sum_i \frac{n_i - 2}{2}\Omega_i \\ &= \frac{1}{2}\Omega_M - \Omega_\Lambda , \end{aligned} \tag{2.8}$$

assuming that the universe is dominated by Ω_M and Ω_Λ. Therefore a positive Ω_Λ tends to accelerate the expansion of the universe while a negative Ω_Λ tends to decelerate it.

By equations (1.13) and (2.6), $\Lambda = 0$ if and only if $\Omega_\Lambda = 0$. The dynamics of universes with $\Omega = \Omega_M + \Omega_\Lambda$ is given in Figure 1 (see Appendix), taken from Carroll's paper. There are three stationary points, namely, (Ω_M, Ω_Λ)=(0,0), (0,1), and (1,0). (0,0) corresponds to an empty universe, while (0,1) corresponds to a universe with no matter density and dominated by the cosmological constant, known as the de Sitter space. (1,0), corresponding to a universe with cosmological constant=0, is called the Einstein –de Sitter solution. It is proposed that the inflationary scenario provides a mechanism driving the universe to the line $\Omega_M + \Omega_\Lambda = 1$, and observations suggest that our universe lies somewhere near the point (0.3,0.7).

3. Observational Test

3.1 Type Ia Supernovae

Astronomers measure distance in terms of the "distance modulus" $m - M$, where m is the apparent magnitude and M is the absolute magnitude of the source. The High-Z Supernova Team and the Supernova Cosmology Project, measuring $m - M$ vs z (the redshift), were able to estimate limits on Ω_M and Ω_Λ. Both teams' results suggested a positive cosmological constant and were inconsistent with an open universe with $\Lambda = 0$, given our current knowledge of the matter density of the universe.

3.2 Cosmic Microwave Background

The COBE satellite has discovered temperature anisotropies in the CMB. Measuring the "Doppler peak" (an increase in power due to acoustic oscillations) of the CMB gave us some hints on the cosmic energy density. Measurements done in 1997 and 1998 on the CMB, combined with those obtained from the supernovae, suggest that our universe lies in the vicinity of $(\Omega_M, \Omega_\Lambda) = (0.3, 0.7)$.

4. Physics Issues: Dark Energy

Although the cosmological constant fits well with our current data, our observations can instead be explained by some form of "dark energy". We can parametrize a component X due to this dark energy by an effective equation of state $p_X = w_X \rho_X$, which is similar to the one we had. Current observations of supernovae, large-scale structure, gravitational lensing and the CMB already give us limits on w_X. The simplest physical model for a dark energy component is a "slowly-rolling" scalar field. However, there are other models; for example, there is a model in which the masses of the dark matter particles increase as the universe expands. The cosmological consequences, however, are difficult to analyze analytically.