Elements of Relativistic Stellar Analysis

Mohammad H. Hamidian

April 12, 2002

1 Introduction

1.1 Overview

While *general relativity* is a remarkable and elegant theory in its own right, its awesome power is realized when it makes contact with the real physical world. From gravitational waves to stellar evolution, Einstein's theory is a landmark of ingenuity, inspiration, and physical rigor. Among some of the great astrophysical theories is that of *relativistic stars*. While classical theories have done great justice to the subject, general relativity gives a more precise account of the events concerning stellar phenomena. Suitable integration with statistical mechanics, creates stellar models which define the birth, evolution, and death of stars. In this article we will examine some simple stellar models of *static* and *stationary* stars. As with the tradition of theoretical physics we begin with the most simple models and add elements of realism so that we may begin to make contact with the least complex systems.

1.2 Newtonian Perspective

Newton's theory of gravity claims that any piece of matter attracts another piece of matter with a force which is inversely proportional to the square of the distance separating them and proportional to the product of their masses. The constant of proportionality is G, Newton's gravitational constant. For convenience with adopt a system of units where G=1. From the inverse square law it can be deduced that the gravitational field of a mass is defined by a single function Φ of position, and possibly of time, which satisfies *Poisson's equation* inside matter:

$$\nabla^2\Phi \equiv \Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\Big)\Phi(x,y,z,t) = -4\pi\epsilon(x,y,z,t), \tag{1}$$

where ϵ is the mass density. Outside matter, Φ satisfies *Laplace's equation*:

$$\nabla^2\Phi = 0, \tag{2}$$

the solution of which is by definition a harmonic function. The function Φ is referred to as the gravitational potential and has the physical significance that a particle of mass m placed in a gravitational field experiences a force $\mathbf{F}$ given by

$$\mathbf{F} = m\nabla\Phi. \tag{3}$$

Since we will interested later in axial or cylindrical symmetries we need to make some clarifications. We first define cylindrical polar coordinates (ρ, ϕ, z) as follows:

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z.$$

Here ρ is the distance of the point P from the z-axis and ϕ is the angle which the plane containing the z-axis and the point P make with the plane $y = 0$. The angle ϕ is referred to as the azimuthal angle. A scalar function $f(x, y, z)$ is said to be axially symmetric (with the z-axis as the axis of symmetry) if, when expressed in terms of polar coordinates, it is independent of the azimuthal angle, ϕ. Thus axially symmetric functions have rotational symmetry about the z-axis. Cylindrical symmetry can be defined is a similar manner if we make the scalar functions independent of z in addition to being independent of ϕ.

We will be partly concerned with rotating systems and thus it is pertinent to consider the simple rotating system as described by Newtonian gravity, specifically the case of uniformly rigidly rotating inviscid homogeneous fluid where the rotation is independent of time. It is well known that the boundary of such a fluid mass is an oblate spheroid. A typical portion of the material of the fluid mass is kept in equilibrium by gravitational, pressure, and centrifugal forces. We will not immediately concern ourselves with the equations governing these forces. If we assume the centre of the spheroid to be at the centre of the coordinate system, the gravitational potential inside the matter is given by

$$\Phi(\rho, z) = a\rho^2 + bz^2 + \Phi_0, \tag{4}$$

where a, b, Φ_0 are constants. From (1) it is seen that a, b, and ϵ are related by

$$\nabla^2\Phi = \Phi_{\rho\rho} + \rho^{-1}\Phi_\rho + \Phi_{zz} = 4a + 2b = -4\pi\epsilon. \tag{5}$$

We will now briefly describe one of the fundamental differences between a Newtonian rotating system and one as described by general relativity so that a realization of the differences in the theories can begin. Let a test particle of mass m be released from rest at great distance from the rotating mass considered above the equatorial plane $z = 0$. According to (3), the force on the test particle is given by

$$\mathbf{F} = \left\{ m\left(\mathbf{e}_\rho \frac{\partial}{\partial\rho} + \mathbf{e}_\phi \rho^{-1} \frac{\partial}{\partial\phi} + \mathbf{e}_z \frac{\partial}{\partial z}\right)\Phi \right\}_{z=0} \tag{6}$$

where $\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z$ are the coordinate axes in cylindrical coordinates. Since the system is axially symmetric about the z-axis, Φ is independent of ϕ so that the coefficient of $\mathbf{e}_\phi$ in (6) vanishes. This is true for all z. Thus for any position of the test particle there is no transverse force on the particle in the $\mathbf{e}_{phi}$ direction. Furthermore, it can be seen that the system has reflection symmetry about the plane $z = 0$, so Φ depends on z and z^2 only (note that (4) is the interior potential only), so that

$$\left\{\frac{\partial\Phi}{\partial z}\right\}_{z=0} = \left\{2z\frac{\partial\Phi}{\partial z^2}\right\}_{z=0} = 0 \tag{7}$$

Thus

$$\mathbf{F} = m\frac{\partial\Phi}{\partial\rho}(\rho, 0)\mathbf{e}_\rho, \tag{8}$$

which means that the test particle experiences a radial force and will travel a path which goes through the centre of the mass. Taking the same situation from a general relativistic perspective. The concept of the equatorial plane and the fact that the system has reflection symmetry can be carried over. One can then ask what would happen to a test particle release from rest on the equatorial plane. In this case there will be a transverse force because of the rotation of the central mass and the particle will follow and curve path away from the equatorial plane. This phenomenon is related to *inertial dragging*. From this one observes that in general relativity matter in motion exerts a force in a similar manner magnetic forces are exerted by electric charges in motion. This is not true in Newtonian gravitation.

2 Static Stars

2.1 Static Spherically Symmetrical Spacetimes

The study of static stars requires the study of spherically symmetric systems and thus there is a natural choice of coordinates: spherical polar coordinates, (r, θ, ϕ) with

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta \quad r = \sqrt{x^2 + y^2 + z^2}$$

where ϕ is the azimuthal angle and θ is polar angle. With these coordinates, the line element in Minkowski space can be written

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \qquad d\Omega^2 = (d\theta^2 + \sin^2\theta d\phi^2). \tag{9}$$

Each surface of constant r and t is a two-sphere. Now consider the sphere r, which has a (θ, ϕ) coordinate system. The line $\theta = const., \phi = const.$ is orthogonal to the two-sphere and has the tangent vector $\mathbf{e}_r$. Since the vector $\mathbf{e}_\phi$ and $\mathbf{e}_\theta$ lie on the sphere, this requires that $\mathbf{e}_r \cdot \mathbf{e}_\theta = \mathbf{e}_r \cdot \mathbf{e}_\phi = 0$. This in turn gives us $g_{12} = g_{13} = 0$. Furthermore, the only spheres are not those of $t = const.$ but all of spacetime. This results is the line $r = const., \theta = const., \phi = const.$ being orthogonal to the two-spheres and gives further restrictions: $g_{02} = g_{03} = 0$. We are, then, finally led to the general metric of a spherically symmetric spacetime:

$$ds^2 = -g_{00} dt^2 + 2g_{01} dr dt + g_{11} dr^2 + r^2 d\Omega^2, \tag{10}$$

with g_{00}, g_{00}, g_{11} as functions of r and t only.

We now wish to incorporate the static notion into our metric. We first define a *static* spacetime to be one in which we can find coordinates t such that all the metric components are independent of t and the geometry is unchanged by time reversal, $t \to -t$. When we move to *stationary* systems, the second condition will not apply. Here, however, it has the following implications when considering a coordinate transformation $\Lambda : (t, r, \theta, \phi) \to (-t, r, \theta, \phi)$: $\Lambda^{\overline{0}}{}_0 = -1$, $\Lambda^i{}_j = \delta^i{}_j$. The new components of the metric read as such

$$g_{\overline{00}} = (\Lambda^0{}_{\overline{0}} g_{00})^2 = g_{00}$$

$$g_{\bar{0}\bar{r}} = \Lambda^0{}_{\bar{0}}\Lambda^1{}_{\bar{r}}g_{01} = -g_{01} \tag{11}$$
$$g_{\overline{11}} = (\Lambda^1{}_{\bar{r}}g_{00})^2 = g_{11}$$

Since the geometry must be unchanged we further derive that $g_{01} = 0$ and finally that

$$ds^2 = -e^{2\Phi}dt^2 + e^{2\Lambda}dr^2 + r^2 d\Omega^2 \tag{12}$$

where $\Phi(r)$ and $\Lambda(r)$ have been introduce with the assumption that $g_{00} < 0$ and $g_{11} >$ everywhere. These assumptions will be shown to be reasonable within stars. We impose one further restriction. Since stars are bounded systems, we can demand that a great distances from the star the geometry approaches flatness. This imposes the boundary conditions

$$\lim_{r\to\infty} \Phi(r) = \lim_{r\to\infty} \Lambda(r) = 0. \tag{13}$$

2.2 Description of Matter Inside a Star

To a high degree of precision, the matter inside any star is a perfect fluid. In such a fluid shear stresses are negligible and energy transport is negligible on a "hydrodynamic time scale". Thus, when constructing models one uses perfect fluid parameters:

$$\rho = \rho(r) = density\ of\ mass - energy\ in\ rest - frame\ of\ fluid;$$
$$p = p(r) = isotropic\ pressure\ in\ rest\ frame\ of\ fluid;$$
$$n = n(r) = number\ density\ of\ baryons\ in\ rest\ frame\ of\ fluid;$$
$$u^\mu = u^\mu(r) = 4 - velocity\ of\ fluid;$$
$$T^{\mu\nu} = (\rho + p)u^\mu v^\nu + pg^{\mu\nu} = stress - energy\ of\ fluid. \tag{14}$$

In order for the star to be static, each element of the fluid must remain at rest in the static coordinate system; i.e., each element must move along a world line of constant r, θ, ϕ; i.e., each element must have 4-velocity components

$$u^1 = \frac{dr}{d\tau} = 0, \quad u^2 = \frac{d\theta}{d\tau} = 0, \quad u^3 = \frac{d\phi}{d\tau} = 0 \tag{15}$$

The normalization of the 4-velocity

$$-1 = g_{\mu\nu}u^\mu u^\nu = g_{00}u^0u^0 = -e^{2\Phi}u^0u^0,$$

wthen determines u^0:

$$u^0 = \frac{dt}{d\tau} = e^{-\Phi} \tag{16}$$

This, together with equation (12) and (14), determine $T^{\mu\nu}$:

$$T^{00} = \rho e^{-2\Phi}, \quad T^{11} = pe^{-2\Lambda}, \quad T^{22} = pr^{-2}\sin^{-2}\theta, \quad T^{\alpha\beta} = 0 \; if \; \alpha \neq \beta. \tag{17}$$

It should be noted, however, that normally one cannot deduce p and ρ from a knowledge of only n. One must also have information about the temperature, T, or the entropy per baryon, s. Then, the laws of thermodynamics in addition to the equations of state will determine the remaining thermodynamics variables. To make the connection between variables which do not require the extra entropy or temperature parameters, information about the star's thermal properties must be obtained. In particular, how energy generation and heat flow associate to distribute the entropy.

In order to eventually arrive at the structure of a star we apply the conservation law as prescribed by $T^{\mu\nu}$:

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{18}$$

From (18) we come into possession of four equations. However, after carrying out all the covariant differentiation one finds that the only equation which is not vacuous is the case when $\mu = 1 (= r)$. The following equation is the result:

$$(\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr}. \tag{19}$$

One interprets this result as the the pressure gradient required to keep the fluid static in the gravitational field, where the effect of the field is reliant on the left side of (19).

The only part of puzzle not yet used are Einstein's field equations:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \equiv G^{\mu\nu} = 8\pi T^{\mu\nu} \tag{20}$$

Using the definition of the Einstein tensor (and naturally that of the Riemann, and Ricci tensors) as well as the metric defined in (12) one arrives at the components:

$$G_{00} = \frac{1}{r^2}e^{2\Phi}\frac{d}{dr}[r(1 - e^{-2\Lambda})], \tag{21}$$

$$G_{11} = -\frac{1}{r^2}e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\frac{d\Phi}{dr}, \tag{22}$$

$$G_{22} = r^2 e^{-2\Lambda}\Big[\frac{d^2\Phi}{dr^2} + \Big(\frac{d\Phi}{dr}\Big)^2 + \frac{1}{r}\frac{d\Phi}{dr} - \frac{d\Phi}{dr}\frac{d\Lambda}{dr} - \frac{1}{r}\frac{d\Lambda}{dr}\Big], \tag{23}$$

$$G_{33} = \sin^2\theta G_{22}. \tag{24}$$

Equating the two sides of the field equations will yield the structural equations we are looking for. Substituting (21) and the T_{00} term from (17) into the field equation, (20) then yields

$$G_{00} = r^{-2}\frac{d}{dr}[r(1 - e^{-2\Lambda})] = 8\pi T_{00} = 8\pi\rho.$$

This equation becomes easy to solve once one notices that it is a linear differential equation in the quantity $e^{-2\Lambda}$. We make a substitution for the term $r(1 - e^{-2\Lambda})$:

$$2m(r) \equiv r(1 - e^{-2\Lambda}); \qquad e^{2\Lambda} = \Big(1 - \frac{2m}{r}\Big)^{-1}. \tag{25}$$

Given this, we arrive at a simple formula:

$$G_{00} = \frac{2}{r^2}\frac{dm(r)}{dr} = 8\pi\rho. \tag{26}$$

Solving for m(r) produces

$$m(r) = \int_0^r 4\pi r^2 \rho dr + m(0). \tag{27}$$

For our purposes we will want the constant of integration to be zero which will produce a smooth space geometry at the origin. The quantity $m(r)$ defined by (25) with $m(0) = 0$, is a relativistic analog of the "mass-energy inside radius r".

We now look at the 11 component of the field equations:

$$G_{11} = -\frac{1}{r^2}e^{2\Lambda}(1 - e^{-2\Lambda}) + \frac{2}{r}\frac{d\Phi}{dr} = 8\pi T_{11} = 8\pi p.$$

Making the substitution (25), one obtains an expression for the gradient of Φ, also called the source equation for Φ:

$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 p}{r(r - 2m)}. \tag{28}$$

With the help of (19), one then obtains

$$\frac{dp}{dr} = \frac{(p + \rho)(m + 4\pi r^3 p)}{r(r - 2m)}. \tag{29}$$

The result (29) is referred to as the *Oppenheimer-Volkoff (OV)*equation of hydrostatic equilibrium. We now have five equations of structure: those for p and ρ, (25), (27), (28), and the OV equation. These equations relate the five structure functions $\rho, p, n, \Phi, \Lambda$. The 8 eight other fields equations we did not use must be either trivial or must be a consequence of the five equations of structure. Furthermore, to construct a stellar model, we must also find suitable boundary conditions.

2.3 External Geometry: The Scharwzschild Metric

Outside a star the density and pressure vanish. This then reduces our considerations to only Φ and $\Lambda = -\frac{1}{2}\ln(1 - \frac{2m}{r})$. For values values of r greater than the radius R of the star, $m(r) = const$. We denote this constant as M. We then proceed to integrate (28) from the surface of the star to infinity, with $p = 0$ and $M = m$. We must also be sure to include the boundary condition (13). All this yields

$$\Phi(r) = \frac{1}{2}\ln(1 - \frac{2M}{r}) = -\Lambda \qquad r > R \tag{30}$$

Consequently, spacetime geometry outside the star becomes

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + \Big(1 - \frac{2M}{r}\Big)^{-1}dr^2 + r^2 d\Omega^2. \tag{31}$$

Equation (31) is the famous *Schwarzschild* geometry.

2.4 Constructing a Stellar Model

Constructing a stellar model requires most of what we have derived as well as provisions of boundary conditions. To construct a model, we first specify the equations of state and a value for the central pressure i.e., $p_c = p(r = 0)$. A value for Φ at the center of the star is also required. We have already agreed that the mass at the center should be zero. These boundary conditions provide uniquely the solution to the coupled equations, (27)-(29) and the equations of state. Integrating the coupled equation outwards from $r = 0$, one stops when the pressure vanishes. The OV equation ensure that the pressure drops monotonically provided that $\rho \geq 0$ for all $r \geq 0$. Zero pressure indicates that we have reached the surface of the star. Having reached the surface, we then renormalize Φ so that it obeys $\Phi(r = R) = \frac{1}{2}\ln(1 - \frac{2M}{R})$. What results is a relativistic stellar model.

3 Stationary Stars

We proceed differently in this section. We are mainly interested in deriving the metric for a stationary system with the use of new mathematical tools. These allow for a much more rigorous derivation in cases where we are trying to expose symmetries. For what remains, we will be working in polar cylindrial coordinates.

3.1 Killing Vectors

Simplification of metrics with given space-time symmetries are not usually trivial problems. In Newtonian theory, for example, spherical symmetry is defined by a centre and that all equidistant points are equivalent. Carrying this notion over to general relativity does not work. The very notion of distance is defined by a metric. This serves as motivation for finding a coordinate independent method of defining space-time symmetries such as axial symmetry and stationarity. *Killing vectors* aid us in this task.

A metric $g_{\mu\nu}(x)$ is form invariant under a tranformation x^{μ} to $x^{'\mu}$ if $g^{'}_{\mu\nu}(x^{'})$ is the same function of $x^{'\mu}$ as $g_{\mu\nu}(x)$ is of x^{μ}. Concisely,

$$g^{'}_{\mu\nu}(y) = g_{\mu\nu}(y), \qquad \forall\, y. \tag{32}$$

Therefore,

$$g_{\mu\nu}(x) = \frac{\partial x^{'\rho}}{\partial x^{\mu}}\frac{\partial x^{'\sigma}}{\partial x^{\nu}} g^{'}_{\rho\sigma}(x^{'}) = \frac{\partial x^{'\rho}}{\partial x^{\mu}}\frac{\partial x^{'\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x^{'}) \tag{33}$$

Such a tranformation is refered to as an isometry. Now consider an infinitesimal isometry tranformation defined by

$$x^{'\mu} = x^\mu + \alpha\xi^\mu(x), \qquad |\alpha| \ll 1. \tag{34}$$

To first order in α, (33) now reads

$$0 = \frac{\partial\xi^\mu}{\partial x^\rho} g_{\mu\sigma} + \frac{\partial\xi^\nu}{\partial x^\sigma} g_{\rho\nu} + \xi^\mu \frac{\partial g_{\rho\sigma}}{\partial x^\mu}. \tag{35}$$

This can be written is terms of the derivates of the covariant components $\xi_\sigma = g_{\mu\sigma}\xi^\mu$:

$$0 = \frac{\partial\xi_\sigma}{\partial x^\rho} + \frac{\partial\xi_\rho}{\partial x^\sigma} + \xi^\mu \left[\frac{\partial g_{\rho\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho\mu}}{\partial x^\sigma} \right] = \frac{\partial\xi_\sigma}{\partial x^\rho} + \frac{\partial\xi_\rho}{\partial x^\sigma} - 2\xi_\mu \Gamma^\mu_{\rho\sigma}$$

or more compactly,

$$0 = \xi_{\sigma;\,\rho} + \xi_{\rho;\,\sigma} \tag{36}$$

Equation (36) is known as *Killing's equation* and a vector field ξ^μ satisfying it is called a *Killing vector* of the metric $g_{\mu\nu}$. If there exists a solution to (36), then the corresponding vector field represents and infinitesimal isometry of the metric and suggests that the metric has some type of symmetry.

An an example, consider the situation where the metric is independent of one of the four coordinates, say x^0. Thus we have

$$g_{\mu\nu,\,0} = 0. \tag{37}$$

Consider the vector field

$$\xi^\mu = (\xi^0, \xi^1, \xi^2, \xi^3) = (1, 0, 0, 0), \tag{38}$$

so that we have $\xi_\mu = g_{\mu\nu}\xi^\nu = g_{\mu 0}$. Furthermore,

$$\xi_{\mu;\,\nu} + \xi_{\nu;\,\mu} = \xi_{\mu,\,\nu} + \xi_{\nu,\,\mu} - g^{\sigma\lambda}(g_{\sigma\mu,\,\nu} + g_{\sigma\nu,\,\mu} - g_{\mu\nu,\,\sigma})\xi_\lambda$$

$$= g_{\mu 0,\,\nu} + g_{\nu 0,\,\mu} - \xi^\sigma(g_{\sigma\mu,\,\nu} + g_{\sigma\nu,\,\mu} + g_{\mu\nu,\,\sigma})$$

$$g_{\mu\nu,\,0} = 0$$

This shows that if (37) is satisfied, the vector (38) gives a solution to Killing's equation. In relativity, a metic which is independent of $t = x^0$ is said to be staionary. In other words, if the metric admits (38), then (37) is satisfied and the metric is stationary.

Let $\xi^{(1)\mu}$ and $\xi^{(2)\mu}$ be two linearly independent Killing vectors. We define the commutator of these vectors as the vector ζ^{μ} expressed as

$$\zeta^{\mu} = \xi^{(1)\mu}_{;\lambda}\xi^{(2)\lambda} - \xi^{(2)\mu}_{;\lambda}\xi^{(1)\lambda}. \tag{39}$$

The coordinate independent notation is written $[\ \xi^{(1)},\ \xi^{(2)}\ ]$. We will now show that ζ^{ν} is also a killing vector. Begin with

$$\begin{aligned}\zeta_{\mu;\,\nu} + \zeta_{\nu;\,\mu} = {} & \xi^{(1)}_{\mu;\,\lambda;\,\nu}\xi^{(2)\lambda} + \xi^{(1)}_{\mu;\,\lambda}\xi^{(2)\lambda}_{;\,\nu} - \xi^{(2)}_{\mu;\,\lambda;\,\nu}\xi^{(1)\lambda} \\ & -\xi^{(2)}_{\mu;\,\lambda}\xi^{(1)\lambda}_{;\,\nu} + \xi^{(1)}_{\nu;\,\lambda;\,\mu}\xi^{(2)\lambda} + \xi^{(1)}_{\nu;\,\lambda}\xi^{(2)\lambda}_{;\,\mu} \\ & -\xi^{(2)}_{\nu;\,\lambda;\,\mu}\xi^{(1)\lambda} - \xi^{(2)}_{\nu;\,\lambda}\xi^{(1)\lambda}_{;\,\mu}\end{aligned} \tag{40}$$

However, from the fact the $\xi^{(1)}$ and $\xi^{(2)}$ are Killing vectors, and taking the covariant derivartive of Killing's equation we have

$$\xi^{(i)}_{\mu;\,\nu;\,\lambda} + \xi^{(i)}_{\nu;\,\mu;\,\lambda} = 0, \qquad i = 1, 2. \tag{41}$$

Before we continue, we prove a useful relation. Consider the second covariant derivative of a covariant vector V_{λ} :

$$\begin{aligned}V_{\mu;\,\nu;\,\kappa} = {} & \frac{\partial}{\partial x^{\kappa}}V_{\mu;\,\nu} - \Gamma^{\lambda}_{\nu\kappa}V_{\mu;\,\nu} - \Gamma^{\lambda}_{\mu\kappa}V_{\lambda;\,\nu}\& \\ = {} & \frac{\partial^2 V_{\mu}}{\partial x^{\nu}\partial x^{\kappa}} - \frac{\partial V_{\lambda}}{\partial x^{\kappa}}\Gamma^{\lambda}_{\mu\nu} - V_{\lambda}\frac{\partial}{\partial x^{\kappa}}\Gamma^{\lambda}_{\mu\nu} \\ & -\Gamma^{\lambda}_{\nu\kappa}\frac{\partial V_{\nu}}{\partial x^{\lambda}} + \Gamma^{\lambda}_{\nu\kappa}\Gamma^{\sigma}_{\mu\lambda}V_{\sigma} - \Gamma^{\lambda}_{\mu\kappa}\frac{\partial V_{\lambda}}{\partial x^{\nu}} \\ & +\Gamma^{\lambda}_{\mu\kappa}\Gamma^{\sigma}_{\lambda\nu}V_{\sigma}.\end{aligned} \tag{42}$$

The terms involving first and second derivatives of V_{μ} are symmetric in ν and κ, but the terms involving V_{μ} itself contain an antisymmetric part. Combining this with the definition of $R^{\sigma}{}_{\mu\nu\kappa}$ in terms of Christoffel symbols gives

$$V_{\mu;\,\nu;\,\kappa} - V_{\mu;\,\kappa;\,\nu} = V_{\sigma}R^{\sigma}{}_{\mu\nu\kappa}. \tag{43}$$

Returing to our original proof, we find

$$\xi^{(i)}_{\mu;\,\nu;\,\lambda} = \xi^{(i)}_{\mu;\,\kappa;\,\nu} + \xi^{(i)\sigma} R_{\sigma\mu\nu\lambda}, \qquad i = 1, 2. \tag{44}$$

With suitable indicial acrobatics, it can be shown that (41) and (44) produce

$$\xi^{(1)}_{\mu;\,\lambda;\,\nu}\xi^{(2)\lambda} + \xi^{(1)}_{\nu;\,\lambda;\,\mu}\xi^{(2)\lambda} = \xi^{(1)\sigma}\xi^{(2)\lambda}(R_{\sigma\mu\lambda\nu} + R_{\sigma\nu\lambda\mu}) \tag{45}$$

$$\xi^{(2)}_{\mu;\,\lambda;\,\nu}\xi^{(1)\lambda} + \xi^{(2)}_{\nu;\,\lambda;\,\mu}\xi^{(1)\lambda} = \xi^{(2)\sigma}\xi^{(1)\lambda}(R_{\sigma\mu\lambda\nu} + R_{\sigma\nu\lambda\mu}) \tag{46}$$

Substracting (45) from (46), and using the symmetry properties of the Riemann tensor for the last step we deduce that

$$\begin{aligned} &(\xi^{(1)}_{\mu;\,\lambda;\,\nu} + \xi^{(1)}_{\nu;\,\lambda;\,\mu})\xi^{(2)\lambda} - (\xi^{(2)}_{\mu;\,\lambda;\,\nu} + \xi^{(2)}_{\nu;\,\lambda;\,\mu})\xi^{(1)\lambda} \\ &= \xi^{(1)\sigma}\xi^{(2)\lambda}(R_{\sigma\mu\lambda\nu} + R_{\sigma\nu\lambda\mu} - R_{\sigma\mu\lambda\nu} + R_{\sigma\nu\lambda\mu}) = 0. \end{aligned} \tag{47}$$

Looking back at (40), one see that terms involving double covariant derivatives vanish. The other terms can be shown to cancel by using Killing's equation. For example,

$$\begin{aligned} \xi^{(1)}_{\mu;\,\lambda}\xi^{(2)\lambda}_{;\,\nu} &= -\xi^{(1)}_{\lambda;\,\mu}\xi^{(2)\lambda}_{;\,\nu} \\ &= -\xi^{(1)\lambda}_{;\,\mu}\xi^{(2)}_{\lambda;\,\nu} \\ &= +\xi^{(1)\lambda}_{;\,\mu}\xi^{(2)}_{\nu;\,\lambda} \end{aligned} \tag{48}$$

which cancel the last term in (40). This proves that ζ^μ is a Killing vector. Now suppose that there are only n linearly independent Killing vectors $\xi^{(i)\mu}, i = 1, 2, \ldots, n$. Since the commutator of any two of these vectors is still a Killing vector, one must be able to represent it as a linear combination of n Killing vectors. The result immediately follows that

$$\xi^{(i)\mu}_{;\,\nu}\xi^{(j)\nu} - \xi^{(j)\mu}_{;\,\nu}\xi^{(i)\nu} = \sum_{k=1}^{n} a^{ij}_k \xi^{(k)\mu}, \qquad i, j = 1, \ldots, n. \tag{49}$$

3.2 Axially Symmetric Stationary Metric

We assume that the source of the rotating field is a steadily (time-independent) rotating star made of perfect fluid. The star and the field it generates have axial symmetry through the $x^2 = z-$axis which passes through the centre of the star. We now proceed to give a derivation. Some details have been left out but the purpose of exposing the Killing vectors is clear.

We have imposed that the space-time geometry of the star (interior and exterior) be stationary and axially symmetric. In the language of Killing vectors, this requires that the metric possess two linearly independent Killing vectors. The vector ξ is to be time-like everywhere while the vector η is space-like everywhere. As one moves to great spatial distances, one experiences asymptotic flatness and arrive at the following metric:

$$ds^2 = dt^2 d\rho^2 - dz^2 + \rho^2 d\phi^2. \tag{50}$$

At spatial infinity, where we adopt the flat metric (50), we give the vectors ξ and η the following form (with $(x^0, x^1, x^2, x^3) = (t, \rho, z, \phi)$):

$$(\xi^0, \xi^1, \xi^2, \xi^3) = (1, 0, 0, 0), \qquad (\eta^0, \eta^1, \eta^2, \eta^3) = (0, 0, 0, 1) \tag{51}$$

$$(\xi_0, \xi_1, \xi_2, \xi_3) = (1, 0, 0, 0), \qquad (\eta_0, \eta_1, \eta_2, \eta_3) = (0, 0, 0, -\rho^2). \tag{52}$$

We have chosen these vectors because they satisfy Killing equation for the metric in (50). Furthermore, they satisfy the properties that ξ is time-like and η is space-like. There exist other Killing vectors but we have restriced out attention to this particular set. Substituting (51) and (52) into the commutator gives zero at spatial infinity. Make a further assumption that ξ and η are the only Killing vectors of the space-time of the star, not just at spatial inifinity. Then by (49)

$$[\xi, \eta] = a\xi + b\eta. \tag{53}$$

Since the commutator vanished at spatial infinity, and the vectors do not vanish, this implies that $a = b = 0$, and that ξ and η commute everywhere. Following a result from differential geometry, which asserts that if ξ and η commute every, then one can find coordinates t and ϕ such that

$$\xi = \frac{\partial}{\partial t}, \qquad \eta = \frac{\partial}{\partial \phi}. \tag{54}$$

The other two coordinates are ρ and z. With the use of this coordinate system (51) holds but (52) does not. Further, it follows from the discussion just after (37) that

$$\frac{\partial g_{\mu\nu}}{\partial t} = 0, \qquad \frac{\partial g_{\mu\nu}}{\partial \phi} = 0 \tag{55}$$

which produces a metric which is only a function of ρ and z. The next step requires what is called orthogonal transitivity. It asserts the existence of a family of two-dimensional surfaces which are orthogonal to the surfaces arrived at by varying (t, ϕ) and keeping the other coordinates constant. The latter surfaces are surfaces of transitivity of the group of motion associated with the Killing vectors ξ and η. The group of motions of ξ are the transformations $(x^0, x^1, x^2, x^3) \rightarrow (x^0 + t', x^1, x^2, x^3)$ for a given parameter t'. The group of motions of η are $(x^0, x^1, x^2, x^3) \rightarrow (x^0, x^1, x^2, x^3 + \phi')$ for ϕ'. Orthogonal transitivity then implies that there exist coordinate ρ and z such that the vector fields $\chi = \frac{\partial}{\partial z}$ and $\vartheta = \frac{\partial}{\partial \rho}$ are each orthogonal to ξ and η. From this we would, for example, obtain $g_{\mu\nu}\xi^{\mu}\vartheta^{\nu} = 0$. However, $\xi = (1, 0, 0, 0)$ and $\vartheta = (0, 1, 0, 0)$. Consequently, $g_{01} = 0$. Similarly, $g_{02} = g_{13} = g_{23} = 0$. The resulting metric is represented as

$$ds^2 = g_{00}dt^2 + 2g_{03}dtd\phi + g_{33}d\phi^2 + g_{AB}dx^A dx^B, \quad A, B \,\varepsilon\, \{1, 2\}. \tag{56}$$

4 Looking Ahead

We have, of course, only scratched the surface of stellar phenomena. With what we have covered, however, we can start building stellar models with the aid of numberical methods. In fact, we have enough to start describing neutron stars and white dwarves. To make our model more realistic, however, we must also incorporate rotating metrics, which we have briefly described. In addition, once a rotating metric has been found, one can then begin to describe the geodesics in such a system. Their analysis will make clear what was mentioned in the introduction that rotating masses cause inertial dragging. Once we introduce angular momentum into the picture, gravitational waves also emerge and we begin to inch closer to more complex stellar systems.