THE INITIAL VALUE FORMULATION OF GENERAL RELATIVITY

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Abstract. The (Cauchy) initial value formulation of General Relativity is developed, and the maximal vacuum Cauchy development theorem is reviewed. There is a short discussion of the evolution equations and associated guage choices, and global results are mentioned briefly in the conclusion.

1. Introduction and Motivation

General Relativity is a theory relating a Lorentzian space-time metric g on a 4-dimensional manifold $\mathcal{M}$ to the matter content of the manifold. The Einstein equation itself is a second-order partial differential equation in the metric. Therefore, it should be possible to obtain a solution of the equation in some neighbourhood of a subset of $\mathcal{M}$ given certain initial conditions on that subset.

To describe the subset being considered, we need some definitions from causality. The Lorentz metric g on $\mathcal{M}$ divides tangent vectors into three classes, depending on whether their magnitude is positive, negative, or zero. A curve $\gamma \subset \mathcal{M}$ is said to be *spacelike* if $g_{ab}t^a t^b > 0$, *timelike* if $g_{ab}t^a t^b < 0$ and *null* if $g_{ab}t^a t^b = 0$ everywhere along γ where t is the tangent vector to γ. Given a time-orientation of $\mathcal{M}$, we define the *future domain of dependence* of a set $S \subset \mathcal{M}$ as $D^+(S) := \{p \in \mathcal{M} |$ all past-directed inextendible non-spacelike curves through p intersect $S\}$. $D^-(S)$ is defined similarly. The *domain of dependence* of S is defined as $D(S) := D^+(S) \cup D^-(S)$.

Physically, all we can hope to predict given initial conditions on a set $S \subset \mathcal{M}$ are conditions on D(S). Assuming that nothing can travel faster than light, data on S should also be sufficient for this purpose. To simplify the causal analysis, we assume that S is *achronal*; that is, $S \cap J^+(S) = \emptyset$, where $J^+(S) := \{p \in S |\ p$ can be reached from S by a non-spacelike curve$\}$. This follows automatically in a universe where time-travel is impossible. If Σ is a closed achronal set such that $D(\Sigma) = \mathcal{M}$, we say Σ is a *Cauchy surface* for $\mathcal{M}$. In [Wald], it is shown that any such Σ is an embedded submanifold of $\mathcal{M}$ of codimension one. Thus, Σ is a three-dimensional hypersurface. In general, $\mathcal{M}$ may not admit a Cauchy surface, for example if $\mathcal{M}$ contains singularities, but we can always restrict $\mathcal{M}$ so that it does. If $\mathcal{M}$ permits a Cauchy surface, we say it's *globally hyperbolic.*

We thus restrict our attention to globally hyperbolic space-times $\mathcal{M}$ and submanifolds Σ which are Cauchy surfaces for $\mathcal{M}$ on which we define our initial conditions. An important result from causality analysis is given in [Wald]: If $\mathcal{M}$ is a globally hyperbolic space-time, there exists a global time function $f : \mathcal{M} \to \mathbb{R}$ such that each surface of constant f is a Cauchy surface. Thus, $\mathcal{M}$ can be foliated by Cauchy surfaces and the topology of $\mathcal{M}$ is $\mathbb{R} \times \Sigma$ where Σ is any Cauchy surface.

This result suggests that we view the problem as solving for the evolution of a spatial metric h along the gradient of the global time function. Our initial data will consist of a spacelike Cauchy surface Σ, a metric h defined on Σ, and some kind of time derivative of h. In the next section we will clarify what h and it's time derivative are.

2. Differential Geometry of Hypersurfaces

The assumption that Σ is an achronal (hence spacelike) hypersurface of the space-time manifold $\mathcal{M}$ has important consequences arising from the differential geometry.

Assuming Σ is orientable, given a Lorentz metric g on $\mathcal{M}$ we can choose a normal n, i.e. an everywhere timelike vector field n on Σ such that $g_{ab}n^a n^b = -1$ and $g_{ab}n^a v^b = 0\ \forall v \in T\Sigma$. We then define

$$h_{ab} = g_{ab} + n_a n_b \tag{2.1}$$

Evaluated on $T\Sigma$, h is a Riemannian metric. Note that $h^a{}_b$ (where the index is raised by g) is the projection operator on $T\mathcal{M}$ into $T\Sigma$. To see both these claims, if $u \in T\mathcal{M}$ with $u = \sum_{i=0}^3 a_i v^i$, where $v^0 = n$ and $v^i, i = 1,2,3$ are a basis for $T\Sigma$, we have:

$$\begin{aligned}
h_{ab}u^b &= g_{ab}u^b + g_{ac}n^c g_{bd}n^d u^b \\
&= \sum_{i=0}^3 a_i g(v^i, .) + g_{ac}n^c \sum_{i=0}^3 g(n, v^i) \\
&= a_0 g(n, .) + \sum_{i=1}^3 a_i g(v^i, .) + a_0 g(n, .) \\
&= g(\sum_{i=1}^3 a_i v^i, .)
\end{aligned}$$

Since g^{ab} is the inverse operation to the final expression, $h^a{}_b u^b = g^{ac}h_{cb}u^b = \sum_{i=1}^3 a_i v^i$, i.e. $h^a{}_b$ is the projection operator into $T\Sigma$. Also, since by the same calculation the v^0 component of u is irrelevant in $h_{ab}u^a$, we have $h_{ab}u^a u^b = g(\sum_{i=1}^3 a_i v^i, \sum_{i=1}^3 a_i v^i)$, and since the v^0 component is responsible for the Lorentz signature of g, h is Riemannian.

The projection operator allows us to view tensors on $\mathcal{M}$ as tensors on Σ by contracting indices with $h^a{}_b$. For example, if $T^{a_1\ldots a_k}{}_{b_1\ldots b_l}$ is a (k,l)-tensor on $\mathcal{M}$, then $h^{a_1}_{c_1}\ldots h^{a_k}_{c_k} h_{b_1}^{d_1}\ldots h_{b_l}^{d_l} T^{c_1\ldots c_k}{}_{d_1\ldots d_l}$ is a (k,l)-tensor on Σ.

Thus, if D_a denotes the h-compatible derivative operator on Σ, we can verify that

$$D_e T^{a_1\ldots a_k}{}_{b_1\ldots b_l} = h^{a_1}_{c_1}\ldots h_{b_l}^{d_l} h_e{}^f \nabla_f T^{c_1\ldots c_k}{}_{d_1\ldots d_l} \tag{2.2}$$

where ∇_a is the g-compatible derivative operator on $\mathcal{M}$.

We can define a tensor K_{ab} on Σ, the *extrinsic curvature* of Σ, by

$$K_{ab} = h^c{}_a h^d{}_b \nabla_d n_c \tag{2.3}$$

It can be shown that

$$K_{ab} = h^c{}_a h^d{}_b \nabla_d n_c = -\frac{1}{2}\mathcal{L}_n h_{ab} = \frac{1}{2}N^{-1}[\dot{h}_{ab} - D_a N_b - D_b N_a] \tag{2.4}$$

where $\dot{h}$ is the the "time derivative" $\mathcal{L}_t h_{ab}$ of h and N and N_a are the *lapse function* and *shift vector*, to be defined in section 5 below in terms of n and the global time function. The second equality follows from the expansion of Lie derivatives as covariant derivatives and the definition of h, and the third equality follows similarly and from the definitions of N and N_a. The details of the proof take some time and may be found in [Wald], Appendices C and E.

Equation (2.4) shows that K is symmetric, and determines the "time derivative" (i.e. the Lie derivative in the direction of the gradient of the global time function) of h. Thus, the initial data set should consist of a set (Σ, K, h), where Σ is a 3 dimensional Riemannian manifold with metric h and K is a symmetric $(0,2)$-tensor field on Σ. In the next section we'll see that h and K are not independent, but first we'll need two more equations.

Let ${H_{abc}}^d$ denote the Riemann curvature tensor of Σ. By definition, ${H_{abc}}^d \omega_d = D_a D_b \omega_c - D_b D_a \omega_c$. Using the definitions of D_a and K_{ab}, one derives *Gauss' equation*:

$$
{H_{abc}}^d = {h_a}^f {h_b}^g {h_c}^k h^d{}_j {R_{fgk}}^j - K_{ac}{K_b}^d + K_{bc}{K_a}^d \tag{2.5}
$$

Similarly, one can check that *Codacci's equation* holds:

$$
D_a K^a_{\ b} - D_b K^a_{\ a} = R_{cd} n^d h^c{}_b \tag{2.6}
$$

See [Wald] section 10.2 or [H&E] section 2.7 for details of the derivations.

3. The Constraint Equations

The Einstein equation is a second-order partial differential equation containing 10 independent components. However, not all of these components contain information about the time evolution of h. In particular, the four components $G_{ab}n^a$ contain no second order time derivatives of the metric. The Gauss-Codacci equations allow us to show this, and at the same time to derive the four constraints these components impose on initial data sets (Σ, h, K).

The Einstein equation is

$$
G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab} \tag{3.1}
$$

The expression $G_{ab}n^b$ contains 4 components, which can be expressed pointwise by contracting the index a with elements of a basis of $T\mathcal{M}$. If we choose a basis such as the one in the previous section, i.e. $v^0 = n$ and $v^i, i = 1,2,3$ are a basis for $T\Sigma$, we can obtain the four constraints by considering $h^b{}_a G_{bc} n^c$ and $G_{ab}n^a n^b$, where the first expression contains the expressions $v^{(i)a} G_{ab} n^b, i = 1,2,3$ via $h^b{}_a$'s action as a projection.

Evaluating the first expression and using the Codacci equation (2.6), we obtain

$$
8\pi h^b{}_a T_{bc} n^c = h^b{}_a G_{bc} n^c = h^b{}_a R_{bc} n^c = D_b K^b{}_a - D_a K^b{}_b \tag{3.2}
$$

where we have used the fact that $h^b{}_a g_{bc} n^c = 0$, by definition of n and h.

Evaluating the second expression and using the Gauss equation (2.5), we obtain

$$\begin{aligned} 8\pi T_{ab}n^a n^b &= G_{ab}n^a n^b \\ &= R_{ab}n^a n^b + \frac{1}{2}R \\ &= \frac{1}{2}R_{abcd}(g^{ac}+n^a n^c)(g^{bd}+n^b n^d) \\ &= \frac{1}{2}R_{abcd}h^{ac}h^{bd} \\ &= \frac{1}{2}\{H + (K^a{}_a)^2 - K_{ab}K^{ab}\} \end{aligned}$$

That is,

$$8\pi T_{ab}n^a n^b = \frac{1}{2}\{H + (K^a{}_a)^2 - K_{ab}K^{ab}\} \tag{3.3}$$

where the last step follows from the Gauss equation (2.5). We see that equations (3.2) and (3.3) refer only to the initial data on Σ; they are the *constraint equations* which our initial data set (Σ, h, K) must satisfy. If we are solving a system involving matter fields ($T_{ab} \neq 0$), the left-hand sides of (3.2) and (3.3) should be given as part of the initial data set.

4. Harmonic Coordinates

In general, it's impossible to say anything about the existence of solutions to an arbitrary partial differential equation. The Einstein equation (3.1) is not necessarily in any form for which existence results are known. Indeed, by expanding the Ricci tensor $R_{\mu\nu}$ in terms of Christoffel symbols and coordinate derivatives, and further expanding the Christoffel symbols into coordinate derivatives of the metric, we have

$$R_{\mu\nu} = -\frac{1}{2}\sum_{\alpha,\beta} g^{\alpha\beta}\{-2\partial_\beta\partial_{(\nu}g_{\mu)\alpha} + \partial_\alpha\partial_\beta g_{\mu\nu} + \partial_\mu\partial_\nu g_{\alpha\beta}\} + F_{\mu\nu}(g, \partial g) \tag{4.1}$$

where F is a function of components of g and their first derivatives. Contracting to get the Ricci scalar, we get a complicated equation that's not of any particularly well understood form. But because of the general covariance of the Einstein equation, we have the freedom to impose a convenient guage (coordinate system) to change the form of the equation to one for which existence theorems exist.

The simplest such choice of coordinates, due to Choquet-Bruhat (1962), are *harmonic coordinates*, coordinate functions x^μ satisfying $\Box x^\mu := \nabla^a\nabla_a x^\mu = 0$. Expanding this coordinate condition, we can calculate

$$\begin{aligned} 0 = \Box x^\mu &= \nabla_a g^{ab}\partial_b x^\mu \\ &= \sum_{\alpha,\beta}\frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}g^{\alpha\beta}\partial_\beta x^\mu) \\ &= \sum_{\alpha}\frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}g^{\alpha\mu}) \\ &= \sum_{\alpha}(\partial_\alpha g^{\alpha\mu} + \frac{1}{2}g^{\alpha\mu}\sum_{\rho,\sigma}g^{\rho\sigma}\partial_\alpha g_{\rho\sigma}) \end{aligned}$$

where the second and fourth equalities come from the formula for a contracted Christoffel symbol and the formula for the derivative of a determinant. Using this expression, we can cancel most of the second derivatives in (4.1) to calculate R_{ab} in harmonic coordinates:

$$R^H_{\mu\nu} := R_{\mu\nu} + \sum_\alpha g_{\alpha(\mu}\partial_{\nu)}\Box x^\mu = -\frac{1}{2}\sum_{\alpha,\beta} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} + \widehat{F}_{\mu\nu}(g,\partial g) = 0 \tag{4.2}$$

This equation is known as the *reduced Einstein equation.* It's importance comes from a theorem due to Leray (1952) that says that any system of the form

$$g^{ab}(x;\phi_j;\nabla_c\phi_j)\nabla_a\nabla_b\phi_i = F_i(x;\phi_j;\nabla_c\phi_j) \tag{4.3}$$

(where ∇ is any derivative operator and ϕ_j are any unknown functions on $\mathcal{M}$), which has a solution $(\phi_0)_j$, also has a unique solution ϕ_j on a neighbourhood O of Σ for any initial data on a Cauchy surface Σ provided the initial data on Σ are close to the initial data for $(\phi_0)_i$ on Σ, and this solution $g_{ab}(x,\phi_j,\nabla_a\phi_j)$ is globally hyperbolic on O. Equation (4.2) is of this form with $\phi = g$. Note that the reduced Einstein equation presented here is for the vacuum case only. For matter fields that are of the appropriate form, e.g. which keep the reduced Einstein equation in the form (4.3), the rest of the discussion still applies.

To prove local existence of a solution to Einstein's equation, we first use the reduced Einstein equation to prove existence for initial data close to Minkowski space. We choose h and K close to flat on a coordinate neighbourhood U of Σ which satisfy the constraint equations (3.2) and (3.3), and specify $(g_{\mu\nu}, \partial g_{\mu\nu}/\partial t)$ on Σ such that they induce h and K. This leaves freedom in ∂g to ensure that $\Box x^\mu = 0$ on Σ. We can then apply Leray's theorem to produce g on a portion O of $\mathcal{M}$ with U as its Cauchy surface, such that the equation (4.2) holds. Since this solution is given in local coordinates, we can confirm that the harmonic coordinate condition holds throughout O, and therefore g on O is a solution of the Einstein equation.

For initial conditions not close to flat spacetime, it can be shown that by rescaling the initial data we can make it arbitrarily close to flat. The coordinate transformations used in the scaling pass through Einstein's equation, and so a solution generated by rescaling can be transformed to give a solution on the original initial data.

Local uniqueness of the developments generated in harmonic coordinates with respect to arbitrary developments follows from uniqueness in Leray's theorem after an appropriate coordinate transformation. A global development, i.e. one which contains all of Σ can then be constructed by patching local developments using local uniqueness. A Zorn's lemma argument can be used to prove existence of a maximal development of Σ, and its uniqueness can be shown using the Hausdorff property.

Thus, using harmonic coordinates, one can prove the following important theorem, due to Choquet-Bruhat and Geroch (1969):

Theorem 1. *Let (Σ, h, K) be an initial data set with Σ a three-dimensional manifold, h a Riemannian metric on Σ, and K a symmetric $(0,2)$-tensor on Σ, such that h and K satisfy the constraint equations* (3.2), (3.3). *Then (Σ, h, K) has a unique, up to isometry, maximal vacuum Cauchy development. That is, there exists a unique spacetime $\mathcal{M}$ with metric g such that Σ is a Cauchy surface for $\mathcal{M}$,* *the induced metric and curvature on Σ are h and K, g is a solution to the vacuum Einstein equation, and every other such manifold can be embedded isometrically into $\mathcal{M}$. The solution is also stable, in the sense that that g depends continuously on h and K in an appropriate topology.*

As previously mentioned, if the stress-energy tensor T_{ab} is of the appropriate form, this result can be extended to non-vacuum spacetimes. In particular, the Einstein-Klein-Gordon and Einstein-Maxwell equations are of this form ([Wald], p.267). It should be noted that there are other ways of reducing the Einstein equation to prove local existence of solutions. In each case, a similar procedure of checking that the guage choice and constraint equations propagate through the reduced solution is necessary to show that it's a solution of the original equation.

5. The Evolution Equations

The maximal vacuum Cauchy development theorem shows that for any initial data set (Σ, h, K) satisfying the constraint equations, the initial value formulation of General Relativity is well posed. To actually solve the initial value problem, we can take advantage of the existence of a global time function to evolve data on the Cauchy surface with respect to time. Because the normal vector field n introduced on Σ will not, in general, coincide with the time gradient, we define the *lapse function* N and the *shift vector* X by the formula $\partial_t = Nn + X$. We can then derive expressions for the time derivatives of h and K. Starting with h, we have:

$$\begin{aligned}
\mathcal{L}_{\partial_t} h_{ij} &= (\partial_t)^c \nabla_c h_{ij} + h_{cj} \nabla_i (\partial_t)^c + h_{ic} \nabla_j (\partial_t)^c \\
&= (Nn^c + X^c) \nabla_c h_{ij} + h_{cj} \nabla_i (Nn^c + X^c) + h_{ic} \nabla_j (Nn^c + X^c) \\
&= Nn^c \nabla_c h_{ij} + h_{cj} \nabla_i Nn^c + h_{ic} \nabla_j Nn^c + \mathcal{L}_X h_{ij} \\
&= N(n^c \nabla_c h_{ij} + h_{cj} \nabla_i n^c + h_{ic} \nabla_j n^c) + \mathcal{L}_X h_{ij} \\
&= N \mathcal{L}_n h_{ij} + \mathcal{L}_X h_{ij}
\end{aligned}$$

where the fourth equality follows from the Liebnitz rule and the fact that $n^c h_{ac} = 0$. Applying (2.4), we get

$$\mathcal{L}_{\partial_t} h_{ij} = -2NK_{ij} + \mathcal{L}_X h_{ij} \tag{5.1}$$

We can similarly derive an expression for the time evolution of K:

$$\mathcal{L}_{\partial_t} K_{ij} = -\nabla_i \nabla_j N + N(R_{ij} + H_{ij} + trKK_{ij} - 2K_{im}K^m_{\ j}) + \mathcal{L}_X h_{ij} \tag{5.2}$$

Equations (5.1) and (5.2) are known as the *evolution equations.* While the Einstein equation was not used in the derivation of (5.1), it is used in the derivation of (5.2). The form of (5.2) presented here corresponds to the vacuum Einstein equation, and expresses the component of the Einstein equation that was not involved in the constraint equations, $G_{ab}h^a_{\ c}h^b_{\ d}$. (This separation of the Einstein tensor is sometimes referred to as a "3+1 split".)

Note that we have introduced two new objects, X and N, in this formulation. This allows us to rephrase the initial value problem as looking for a map $t \mapsto (h(t), K(t), X(t), N(t))$ on some interval (t_a, t_b), satisfying the evolution equations, with h, K, X, and N specified at some time $t_0 \in (t_a, t_b)$. At each time t, the image of the map can be seen as one leaf of a foliation of $\mathcal{M}$ by Cauchy surfaces. It can be shown that if the initial data satisfies the constraint equations, so does its evolution on each leaf. Since the constraint equations combined with the evolution equations are equivalent to Einstein's equation in a vacuum, the space-time constructed by letting $\mathcal{M} = \Sigma \times (t_a, t_b)$ and $g = -N^2 dt^2 + h_{ij}(dx^i + X^i dt)(dx^j + X^j dt)$ is a solution to the vacuum Einstein equation.

The introduction of X and N do not introduce new freedoms into the system; they just correspond to fixing a guage. Their choice determines the properties of the solution curve. Particular choices of X and N may introduce singularities that are not present in the Cauchy development itself, for example by introducing degenerate foliations. An example is the Gauss foliation condition, $N = 1, X = 0$. In this case the foliation flows in the direction of its normal, and thus, even in Minkowski space a foliation with leaves of non-zero curvature will develop singularities, i.e. when a leaf folds into itself. ([Andersson], 2.2.)

We can describe certain guage choices as *time guages* or *spatial guages* depending on whether they constrain N or X. A variety of time and spatial guages have been studied, and local existence and uniqueness results have been given for some of them ([K&N]). A particularly interesting time guage is the *constant mean curvature*, or CMC, guage, which requires that the leaves of the foliation have constant mean curvature, i.e. $trK = c_t$, where c_t depends only on the global time function. If $\mathcal{M}$ is globally hyperbolic and satisfies the physically plausible *timelike convergence criterion* $R_{ab}V^aV^B$ for all timelike V, then there exists at most one Cauchy surface with a given mean curvature. If there are sequences of Cauchy survaces with mean curvature tending uniformly to $+-\infty$, $\mathcal{M}$ is said to have *crushing singularities.* If $\mathcal{M}$ satisfies the timelike convergence criterion and has crushing singularities, then $\mathcal{M}$ can be foliated globally by CMC hypersurfaces. In this case it makes sense to set $c_t = t$. Using the evolution and constraint equations we can then derive $-\Delta N + |K|^2 N = 1$ which shows that the CMC guage is a time guage.

A CMC foliation may imply statements about the beginning and end of the universe. A mean curvature of $-\infty$ in the past corresponds to a "big bang", and a curvature of $+\infty$ in the future corresponds to a "big crunch". A 3-manifold has *Yamabe type* -1 if it admits no Riemannian metric with scalar curvature $R = 0$ (which implies no metric of positive scalar curvature), type 0 if it admits a metric with $R = 0$ but no metric with $R = 1$ and type $+1$ if it admits a metric with $R = 1$. Because all leaves in the foliation are diffeomorphic, they all have the same Yamabe type. If Σ is of Yamabe type -1, by (3.3), since $K^{ab}K_{ab} >= 0$ and the left-hand side is positive by the dominant energy condition, $\mathcal{M}$ cannot contain a leaf of mean curvature ($= trK$) 0. This suggests that the interval on which the foliation exists is $(-\infty, 0)$. In the cases of Yamabe type 0 or $+1$, the foliation should exist on $(-\infty, 0)$ and $(-\infty, \infty)$, respectively. These statements, concerning global existence in the CMC guage, are at present conjectures but have been proved under symmetry assumptions. Even if the conjecture holds, it may not be the case that the CMC foliation covers $\mathcal{M}$, though type $+1$ foliations will. ([Andersson], [K&N])

6. Conclusion

In general, the question of global existence of solutions to the Cauchy initial value problem is still an open problem. The two general approaches are to restrict to symmetrical spacetimes or to show existence for small data, that is, data close to initial data for which solutions are known (i.e. to show stability of a known spacetime). There are various results for different types of symmetries. A symmetry of a manifold $\mathcal{M}$ is an isometry $\mathcal{M} \mapsto \mathcal{M}$. One type of symmetry is that generated by a Killing field, a vector field on $\mathcal{M}$ whose flow is an isometry. Symmetries generated by sets of Killing fields can be classified in terms of the dimension of their generating Lie algebra. [Andersson] reviews results for 1, 2, and 3 dimensial algebras. For an example of what these symmetries mean, consider a space with a 3-dimensional isometry group generated by spacelike Killing fields, known as a Bianchi, or spatially homogenous, spacetime. In a Bianchi spacetime, the orbits of the isometry group are three-dimensional Cauchy surfaces. This implies that any point in space can be mapped onto any other point in space isometrically. The existence of global CMC foliations has been proved for certain of these symmetrical spacetimes. For spacetimes with no symmetries, only a handful of small-data results are known (see [Andersson] or [Rendall] for a review, or [K&N] for one such result).

There are also initial value formulations other than the Cauchy one. [Rendall] mentions the characteristic initial value problem, which gives initial data on one or more null (rather than spacelike) hypersurfaces. The advantage is apparently that the constraint equations reduce to ordinary differential equations. A local existence theorem exists for the case where data is given on two null hypersurfaces which intersect transversally on a spacelike hypersurface. Another variant is to specify data on a light cone, but there is no existence theorem for this case.

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