# Black holes

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## 1 Spherically symmetric black holes

### 1.1 Schwarzschild solution

A black hole is a region in space with gravitation so strong that even light cannot escape from it. First, let us consider a black hole in empty space. All static spherically symmetric solutions of *vacuum Einstein equations* were found by Schwarzschild in 1915. In spherical coordinates $(t, r, \theta, \varphi)$, the *Schwarzschild solutions* have the following form

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1.1)$$

where $G$ is Newton's gravitational constant, $c$ is the speed of light, and $M$ is a papameter. This metric is singular at the points where $r = \frac{2GM}{c^2}$ and $r = 0$. The constant $r_g = \frac{2GM}{c^2}$ is called *Schwarzschild* or *gravitational radius*, and the surface $r = r_g$ is called *Schwarzschild sphere*. The spacetime inside the Scwarzschild sphere is an example of a black hole. Let us study this example.

The coordinates $(t, r, \theta, \varphi)$ form the *Schwarzschild reference frame*. This frame is valid for the region outside of the black hole. In the Schwarzschild metric, denote by $g_{00}, g_{11}$ coefficients with $c^2dt^2$ and $dr^2$ respectively.

### 1.2 Radial motion of light

The following equation describes the radial motion of light in the Schwarzschild reference frame

$$\frac{dr}{dt} = \pm c(1 - \frac{r_g}{r}). \quad (1.2)$$

Here $t$ is the time measured by a distant observer. Since light always propagate along null geodesics $ds^2 = 0$, and since motion is radial, $d\theta = d\varphi = 0$. If in equation (1.1) we put $ds = d\theta = d\varphi = 0$, we get equation (1.2). Denote by $d\tau = \sqrt{-g_{00}}dt$ the physical time. When $r$ is close to $r_g$ the physical time is much slower than the coordinate time. What happens with light in the vicinity of the Schwarzschild sphere from the standpoint of a distant observer?

1) Speed of light tends to 0 as $r$ tends to $r_g$.

2) It takes infinite time for a light ray to reach $r_g$ (though the proper time is equal to 0). To find the coordinate time we integrate (1.2).

3) Light becomes blueshifted if it goes away from the black hole, and redshifted if it goes towards the black hole. Indeed, consider light signals emitted radially from the point $r = r_1$ at a coordinate time interval $\Delta t$. Since the spacetime is static, when the signals reach an observer at the point $r = r_2$ the coordinate time interval between them will be the same. The corresponding proper time intervals $\Delta\tau_1, \Delta\tau_2$ are equal to $\sqrt{-g_{00}(r_1)}\Delta t$, $\sqrt{-g_{00}(r_2)}\Delta t$ respectively. If $\omega_1, \omega_2$ are frequencies of these light signals measured at points $r = r_1, r_2$ respectively, then

$$\frac{\omega_1}{\omega_2} = \frac{\Delta\tau_2}{\Delta\tau_1} = \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}} = \sqrt{\frac{1 - \frac{r_g}{r_2}}{1 - \frac{r_g}{r_1}}}. \tag{1.3}$$

## 1.3 Radial motion of particles

Consider a free radial motion of a non-relativistic particle in the Schwarzschild reference frame. Since gravitational field is static, the energy $E = mc^2\sqrt{-g_{00}}/\sqrt{1 - v^2/c^2}$ of the particle is conserved. Here $v$ is the physical velocity of the particle, i.e. $v = \sqrt{g_{11}}dr/\sqrt{-g_{00}}dt$. If the particle is at rest at infinity, then $E = mc^2$. It follows that the free radial motion of this particle is described by the following equation

$$\frac{dr}{dt} = \pm\left(1 - \frac{r_g}{r}\right)\sqrt{\frac{r_g}{r}}c \tag{1.4}$$

Notice that for $r \gg r_g$, this equation takes the form

$$\frac{dr}{dt} = \frac{2GM}{r}.$$

The latter describes the motion of the particle in the Newtonian gravitatinal field created by a spherically symmetric body of mass $M$. Thus the parameter $M$ in the expression for the Schwarzschield metric can be interpreted as the mass of the gravitational source.

The coordinate time (measured by the clock of a distant observer) required for the particle to reach the gravitational radius is infinite, because it was infinite even for light rays. Let us find the corresponding proper time, i.e. time measured by the clock of the particle itself. Since proper time is equal to the length of geodesic given by equation (1.4), we get

$$\Delta T = \frac{2}{3}\frac{r_g}{c}\left(\left(\frac{r_1}{r_g}\right)^{\frac{3}{2}} - \left(\frac{r}{r_g}\right)^{\frac{3}{2}}\right), \tag{1.5}$$

where $\Delta T$ is the proper time of fall from $r_1$ to $r$. For any $r > 0$, in particular, for $r = r_g$, this time is finite. Thus for an observer moving along with the particle it takes finite time to reach the black hole.

## 1.4 Lemaître coordinate frame

In order to study the spacetime within the black hole, we need to choose a reference frame that is nonsingular for $r \leq r_g$. The most natural choice is a reference frame fixed to freely falling particles that have zero velocity at infinity. A new time coordinate $T$ is the time measured by the clock fixed to the falling particle. A new radial coordinate $r_1$ is the coordinate that marks particles and remains unchanged for each of them, i.e. the motion of each particle is given by the equation $r_1 = const$. We still have freedom in choice of $T$ and $r_1$. To fix coordinates $(T, r_1)$ let us identify a point $(0, r_1)$ in the new coordinates with the point $(R, r_1)$ in the old ones, where $R = (2/3)r_g(r/r_g)^{3/2}$ (under these conditions metric (1.6) does not contain term $dTdr_1$). The coordinate $r$ is recovered from $(r_1, T)$ by the equation (1.5). In other words, if a freely falling particle begins the motion at the point $r_1$, it arrives at the point $r$ in the proper time interval $T$. To simplify this relation let us take instead of $r_1$ a new coordinate $R$ written above. Now an equation $r = const$ defines a line $R - cT = const$ in the $(T, R)$-plane. In the new coordinates $(T, R, \theta, \varphi)$ Schwarzschild metric takes the form

$$ds^2 = -c^2 dT^2 + \frac{r_g dR^2}{r} + r(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1.6}$$

$$\text{where } r = r_g^{\frac{1}{3}}(\frac{3}{2}(R - cT))^{\frac{2}{3}}).$$

This metric is nonsingular everywhere except for the points where $r = 0$. The coordinates $(T, R, \theta, \varphi)$ form so-called *Lemaître reference frame.* It is valid in the region $r > 0$. The Schwarzshild sphere in this frame is given by the equation $3/2(R - cT) = r_g$.

## 1.5 Spacetime within the black hole

The world lines of radial light rays in the Lemaître coordinates are described by the following equation

$$c\frac{dR}{dT} = \pm\left(\frac{\frac{3}{2}(R - cT)}{r_g}\right)^{\frac{1}{3}}. \tag{1.7}$$

As shown in Figure 1, the whole future light cone at any point within the black hole lies within the black hole. Thus neither particles nor light can escape from the black hole. They will move towards the singularity at $r = 0$. This is the direction into the future. Equation (1.7) shows that world lines of both outgoing and ingoing light rays reach the singularity at $r = 0$ in finite coordinate time $T$. The Schwarzshild sphere traces an *event horizon* of the black hole.

Notice that lines $r = const$ are timelike if $r > r_g$, lightlike if $r = r_g$ and spacelike if $r < r_g$. This is the reason why spacetime inside the Schwarzschild sphere is no longer stationary, i.e. the "time translation" vector field $\frac{d}{dt}$ is not timelike. The coordinate $-r$ can be chosen as a time coordinate inside the black hole, while $t$ becomes a spatial radial coordinate. Thus we obtain

Figure 1: Black hole in the Lemaître coordinates.

another frame of reference inside the black hole. In this frame, time and spatial directions change their roles.

## 1.6 Kruskal coordinates

A coordinate frame is said to be *complete*, if every geodesic either extends to all values of its natural parameter or arrives to the true physical singularity. A singularity is called *physical* if it can not be removed by any change of coordinates. By this definition both Schwarzschild and Lemaître coordinate frames are incomplete. In the Lemaître coordinates, consider a particle moving freely along the radius away from the Schwarzschild sphere. Its world line continued to the past will reach the event horizon in finite proper time, say $\tau_0$. But the corresponding coordinate time is infinite because no particle can escape from the event horizon. Thus in proper time $\tau_0$ in the past history of the particle, there was no black hole inside the event horizon. An *eternal black hole* does not exist in any complete extension of the Schwarzschild coordinates.

One of the possible complete extensions of the Schwarzschild coordinates is given by the *Kruskal coordinates.* First, let us choose the coordinate system $(u, v)$ fixed to the radial light rays, e.g. $u = ct - r_*, v = ct + r_*$, where $r_* = r + r_g \ln |r/r_g - 1|$ is the so-called *tortoise coordinate.* Second, we reparameterize null geodesics to avoid coordinate singularity at $r = r_g$. Finally, we get Kruskal coordinates $(T, X, \theta, \varphi$ connected to the coordinates $(t, r, \theta, \varphi)$ by the following transformation:

$$\begin{pmatrix} T \\ X \end{pmatrix} = \left| \frac{r}{r_g} - 1 \right|^{1/2} e^{r/2r_g} \begin{pmatrix} \cosh(ct/2r_g) \\ \sinh(ct/2r_g) \end{pmatrix}$$

Figure 2: Spacetime in the Kruskal coordinates.

In these coordinates the metric takes the form

$$ds^2 = \frac{4r_g^3}{r} e^{-r/r_g}(-dT^2 + dX^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

and radial light rays propagate along the lines $(T \pm X) = const$ as in flat space. In the $(T, X)$- plane the curves $r = const$, $t = const$ are respectively hyperbolas $X^2 - T^2 = const$ and lines passing through the origin.

Kruskal spacetime is divided into 4 regions by hypersurfaces $X \pm T = 0$ (see Figure 2). Both regions I and III are isometric to the region of the Schwarzschild spacetime outside of the black hole. Region II is the black hole, while region IV is a *white hole* — the region obtained from the black hole by time reversal. No signals emitted in II can reach the outside regions I or III. The opposite is true for region IV: no signal emitted from I or III can enter it. If we consider sections $T = const > 0$ we obtain two asymptotically flat regions connected by a tunnel inside the black hole. The boundary of the tunnel consists of two event horizons: they look like the black and white holes respectively for observers from regions I and III. Since the global topology is not fixed by Einstein field equations we may identify these regions and obtain a so-called *wormhole* (see Figure 3).

# 2 Hawking radiation

In flat Minkowski space, consider a scalar field $\varphi(x)$ with mass $m$ satisfying the Klein-Gordon equation $(\Box + (m/\hbar)^2)\varphi(x) = 0$. It can be decomposed into positive and negative frequencies according to

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int [a_k e^{i(k,x) - i\omega_k t} + a_k^+ e^{-i(k,x) + i\omega_k t}] \frac{d^3k}{\sqrt{2\omega_k}}, \tag{2.1}$$

Figure 3: Wormhole

where $\omega_k = \sqrt{k^2 + (m/\hbar)^2}$. Here the integration over 4-dimensional space with coordinates $(\omega, k_1, k_2, k_3)$ is reduced to the integration over 3-dimensional hypersurface $\omega^2 - k_1^2 - k_2^2 - k_3^2 = (m/\hbar)^2$ by using the Klein-Gordon equation. Now let us quantize $\varphi(x)$. With every pair of functions $a_k, a_k^+$ we associate a pair of conjugate operators $a_k, a_k^+$ called respectively the annihilation and creation operators. They satisfy the following relations

$$[a_k, a_{k'}^+] = \mathrm{Id}, \text{ if } k = k' \text{ and } 0 \text{ otherwise },$$

$$[a_k, a_{k'}] = [a_k^+, a_{k'}^+] = 0.$$

Let us describe the *Fock space* $V$ of states with the action of these operators. The vacuum state $|0\rangle \in V$ is defined by

$$a_k|0\rangle = 0 \text{ for any } k,$$

and the other states are its images under the action of the creation operators.

Now consider quantum field theory on a globally hyperbolic curved spacetime with metric $g_{ab}$. Denote by $g$ the determinant of $g_{ab}$. The curved-space version of (2.1) is

$$\Box_g \varphi + \frac{m}{\hbar}\varphi = 0, \text{ where } \Box_g = \frac{1}{\sqrt{g}}\partial_a[\sqrt{g} g_{ab} \partial_b]. \tag{2.2}$$

Consider now two solutions $u_1, u_2$ of this equation and define their inner product by the following formula

$$(u_1, u_2) = i \int_\Sigma (\bar{u}_1 \nabla_a u_2 - u_2 \nabla_a \bar{u}_1) n^a dV = (\bar{u}_2, \bar{u}_1).$$

Here $n^a$ denotes the normal vector with respect to some spacelike hypersurface $\Sigma$. The Klein-Gordon equation guarantees that this inner product is independent of the choice of $\Sigma$. Choose an orthonormal basis $u_k, \bar{u}_k$ of solutions. Note that in Minkowski space there always exists a distinguished basis, namely, eigenbasis for the momentum operator. Any field $\varphi$ can be expanded into the chosen basis

$$\varphi(x) = \int (a_k u_k + a_k^+ \bar{u}_k) d\mu(k),$$

where $\mu(k)$ is the used measure. As in the flat case a Fock space can be constructed from the vacuum state $|0\rangle_u$ that is normed by condition ${}_u\langle 0|0\rangle_u = 1$. The crucial difference between flat and curved cases is that in a general spacetime there is no distinguished basis. Thus vacuum state $|0\rangle_u$ depends on the chosen set of solutions $u$.

Therefore, one can expand the field $\varphi$ into a different orthonormal basis $\{v_p, \bar{v}_p\}$

$$\varphi(x) = \int (b_p v_p + b_p^+ \bar{v}_p) d\mu(p).$$

One can also expand one basis into the other

$$v_p = \int (\alpha(p,k) u_k + \beta(p,k) \bar{u}_k) d\mu(p),$$

where $\alpha$ and $\beta$ are the so-called *Bogolubov coefficients*:

$$\alpha(p,k) = (u_k, v_p) \qquad \beta(p,k) = -(\bar{u}_k, v_p).$$

Then the 'old' creation and annihilation operators are expressed in terms of the 'new' ones

$$(a_k, a_k^+) = \int (b_p, b_p^+) \begin{pmatrix} \alpha(p,k), & \beta(p,k) \\ \bar{\beta}(p,k) & \bar{\alpha}(p,k) \end{pmatrix} d\mu(p).$$

The operator $a^+ a$ 'measures' the particle content of type $k$ in a given state and is called the *particle number operator*. Its expectation value $\langle 0|a^+ a\|0>$ with respect to the vacuum state is of course zero. However, the expectation value of the 'new' particle number operator $b^+ b$ with respect to the 'old' vacuum state $|0\rangle_u$ does not vanish in general:

$${}_u\langle 0|b_p^+ b_p|0\rangle_u = \int |\beta(p,k)|^2 d\mu(k).$$

Thus the 'old' vacuum contains 'new' particles!

However, for the static spacetime there exists a distinguished set of solutions $u_k$ with positive frequencies that satisfies the equation

$$\frac{\partial u_k}{\partial t} = -i\omega_k u_k.$$

If a different set of solutions $\{v_p\}$ is a linear combination of the $\{u_k\}$ only, the Bogolubov coefficient $\beta(p,k)$ is zero and both sets of solutions share a common vacuum.

Let us study an important example of particle creation in flat spacetime. This example is closely related to particle creation of black holes. Consider a uniformly accelerated observer in 2-dimensional Minkowski spacetime with coordinates $T, X$. The observer moves along the hyperbola $X^2 - T^2 = a^{-2}$, where $a$ is the norm of the proper acceleration. In coordinates $(\tau, \rho)$, s.t.

$$\begin{pmatrix} T \\ X \end{pmatrix} = \rho \begin{pmatrix} \cosh(a\tau) \\ \sinh(a\tau) \end{pmatrix},$$

all such observers are static. The coordinates $(\tau, \rho)$ describe the so-called *Rindler spacetime* with the metric

$$ds^2 = dT^2 - dX^2 = a^2\rho^2 d\tau^2 - d\rho^2. \tag{3.3}$$

This spacetime has coordinate singularity at $\rho = 0$. Thus the accelerated observer in flat Minkowski space encounters a horizon, although there is no singularity behind this horizon. The Rindler space is analogous to the Schwarzschild space in Kruskal coordinates. The metric (3.3) is very similar to Schwarzschild metric in the vicinity of the event horison

$$ds^2 \approx \kappa\rho^2 dt^2 - d\rho^2 - \frac{1}{4\kappa^2}(d\theta^2 + \sin^2\theta d\varphi^2),$$

where $\rho^2 = 4r_g(r - r_g)$ and $\kappa = 1/2r_g$ is the so-called *surface gravity.* First two terms on the right-hand side correspond exactly to the metic (3.3) with the acceleration $a$ replaced by the surface gravity $\kappa$. Consider a massless scalar field $\varphi$ in Minkowski spacetime. An inertial observer has a distinguished set of solutions $\{u_k\}$ consisting of plane waves

$$u_k(T, X) = \frac{1}{\sqrt{4\pi|k|}} e^{-i|k|T + ikX}.$$

Thus $\varphi$ is quantized according to (2.1)

$$\varphi(T, X) = \int (a_k u_k + a_k^+ \bar{u}_k) dk$$

The accelerated observer, however, has another disinguished set of solutions $\{v_p\}$ according to (2.2) for the Rindler metric

$$v_p(\tau, \rho) = \frac{1}{4\pi|p|} e^{-i|p|\tau} \rho^{ip/a},$$

$$\varphi(\tau, \rho) = \int (b_p u_p + b_p^+ \bar{v}_p) dp.$$

Calculating the Bogolubov coefficient $\beta(p, k) = -(\bar{u}_k, v_p)$, one finds that the expectation value of the particle number operator $b_p^+ b_p$ with respect to the standard Minkowski vacuum $|0\rangle_M$ is equal to

$${}_M\langle 0|b_p^+ b_p|0\rangle_M = \int |\beta(p, k)|^2 dk = (volume) \times \frac{1}{e^{2\pi|p|/a} - 1}.$$

This equation describes a Planckian distribution at a temperature

$$T = \frac{\hbar a}{2\pi k_B}, \tag{2.4}$$

where $k_B$ is the Boltzmann constant. Thus a uniformly accelerated observer sees a thermal distribution of particles in Minkowski vacuum. This demonstrates that vacuum state is not unique even for flat spaces.

Since Rindler metric and Schwarzschild metric are very sumilar near the horison, it is natural to expect that according to the equivalence principle, there is a black hole radiation with temperature (2.4), where $a$ is replaced by $\kappa$. Indeed, it is true. It was proved by Hawking in 1975 that the black hole produces particles according to the Plank distribution with the temperature

$$T_{BH} = \frac{\hbar \kappa}{2\pi k_B}.$$

This effect is called *Hawking radiation.*

# References


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