An invariant tensor in $S^3(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{sl}(n)$, $n \geq 3$

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In this short note we describe a simple construction of a non-zero $\mathfrak{g}$-invariant tensor in $S^3(\mathfrak{g})$, where $\mathfrak{g} = \mathfrak{sl}(n)$, $n \geq 3$. We learned of this construction from Eric Weinstein (with the help of Dror Bar-Natan). Since elements in $S^3(\mathfrak{g})$ are antipodally odd and the PBW isomorphism $S(\mathfrak{g})^{\mathfrak{g}} \cong U(\mathfrak{g})^{\mathfrak{g}}$ is antipode preserving, the Weinstein's tensor provides a counterexample to the author's unmotivated guess that the elements of $U(\mathfrak{g})^{\mathfrak{g}}$, for $\mathfrak{g}$ semisimple, are antipodally even.

Let $\mathfrak{g}$ be a classical simple complex Lie algebra in its standard representation as a Lie subalgebra of $\mathfrak{gl}(m)$, for some m. We endow the latter Lie algebra with the invariant metric $(X, Y) = \mathrm{tr}(XY)$. Recall that this metric remains non-degenerate when restricted to $\mathfrak{g}$. We use the induced metric on $\mathfrak{g}$ to identify $\mathfrak{g} \cong \mathfrak{g}^*$.

We define the following tri-linear form on $\mathfrak{g}$:

$$(X, Y, Z) \longrightarrow (XY + YX, Z), \qquad X, Y, Z \in \mathfrak{g}\,.$$

This form is symmetric (because $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$) and $\mathfrak{g}$-invariant. (For example, take G to be the compact Lie group corresponding to the compact real form of $\mathfrak{g}$ and use $\mathrm{tr}(gAg^{-1}) = \mathrm{tr}(A), g \in G$, to get G-invariance, which implies $\mathfrak{g}$-invariance.)

So we have the promised invariant symmetric tensor *provided* our tri-linear form is not identically zero. This is the case for $\mathfrak{g} = \mathfrak{sl}(n)$, $n \geq 3$: Take $X = Y$ having their upper left 2×2 block equal to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and all other entries are zero. Take Z to be the diagonal matrix $\mathrm{diag}(1, 0, \ldots, 0, -1)$. We have $(X, Y, Z) \to 2 \neq 0$, as desired.

We now demonstrate that the construction "fails" for the other classical algebras — the symplectic and orthogonal Lie algebras (hence also for $\mathfrak{sl}(2) \cong \mathfrak{sp}(1)$). We only need to know that each of these algebras is defined as the set of matrices $X \in \mathfrak{gl}(m)$ satisfying

$$X^t A + AX = 0\,, \tag{1}$$

where A is a certain invertible matrix. Simple computations yield the following two results:

- If X, Y satisfy (1), then $Z = XY + YX$ satisfies

$$Z^t A - AZ = 0\,, \tag{2}$$

- If X satisfies (1) and Z satisfies (2), then $(ZX)^t = -A(XZ)A^{-1}$, hence

$$\mathrm{tr}(XZ) = 0\,.$$

Clearly, these results complete our demonstration.

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