Elevating link homology theories and TQFT’es via infinite cyclic coverings

Oleg Viro

May 27, 2011

Introduction

Results. 1

New invariants related to old ones.

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Immersed surfaces in S^4 transversal to a standardly embedded S^2.

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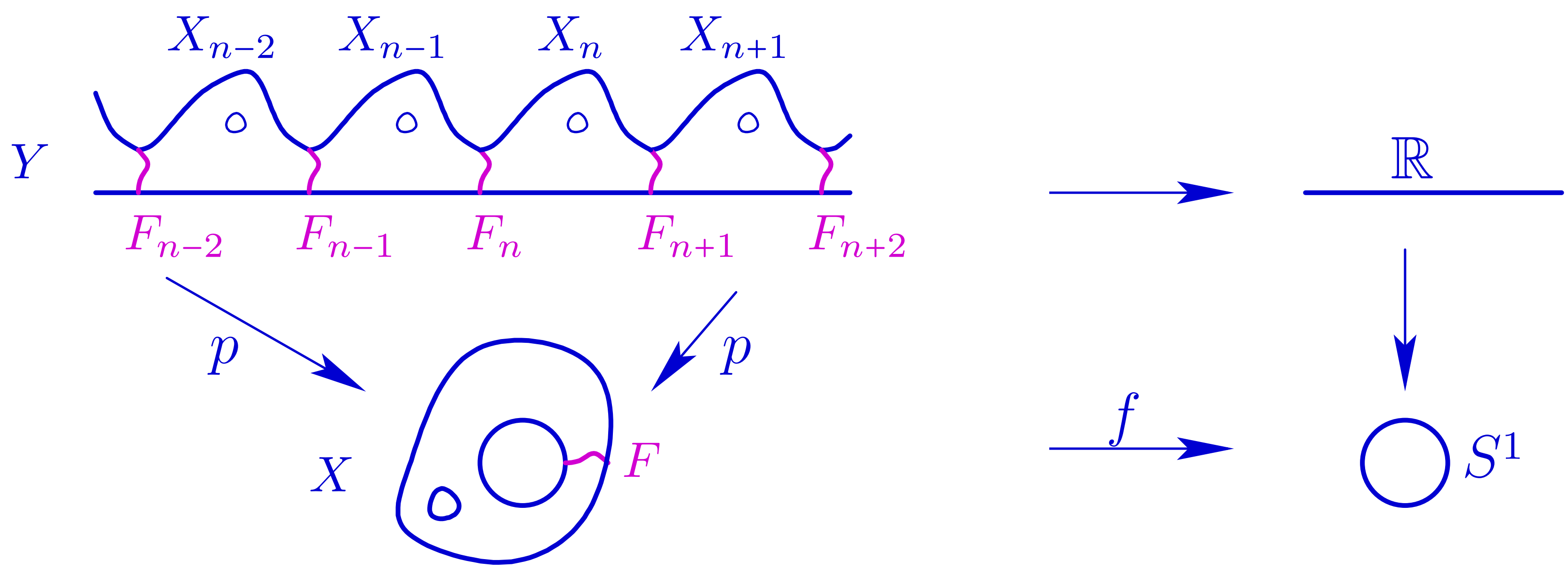

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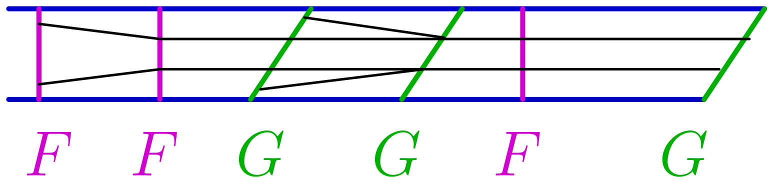

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If $X = S^3 \smallsetminus K$, $Z(F) = H_1(F;\mathbb{Q})$, then
this is Seifert's calculation of the Alexander module $H_1(Y;\mathbb{Q})$ of K.

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For 3-manifolds and various TQFT's, it was studied by Pat Gilmer in 90s.

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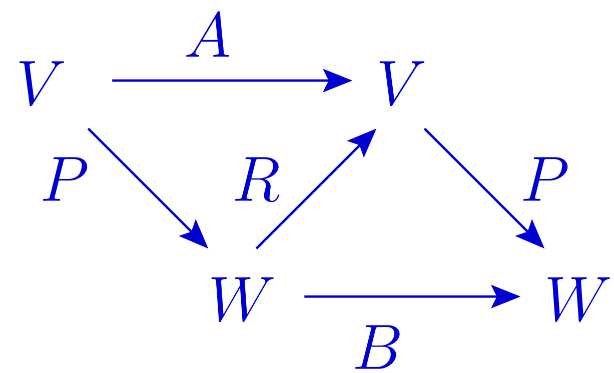

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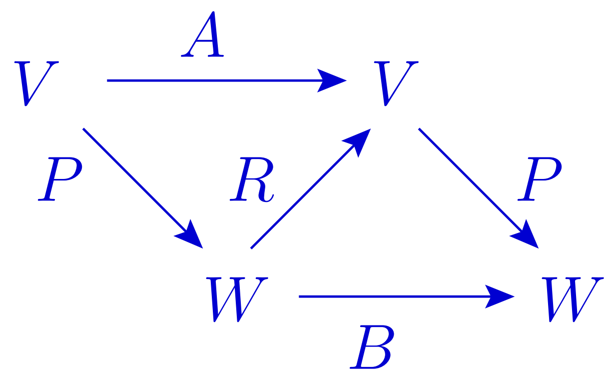
Diagram  commutes.

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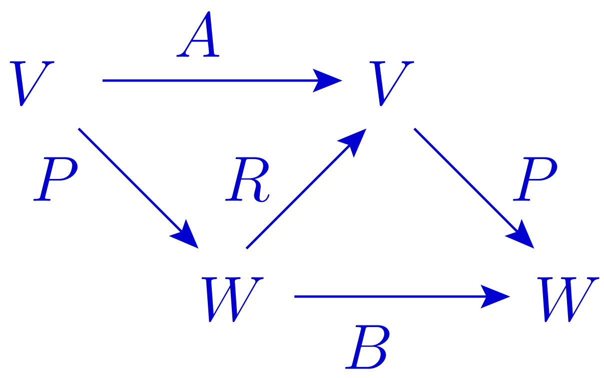

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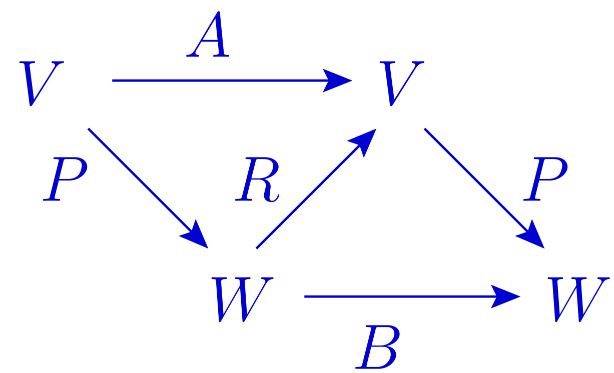
Diagram 

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Theory of Skeletons

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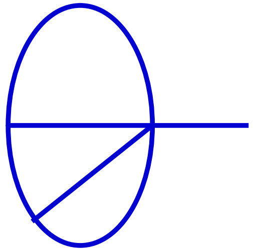

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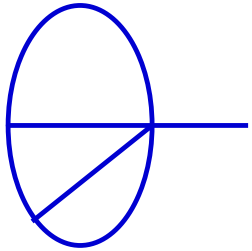

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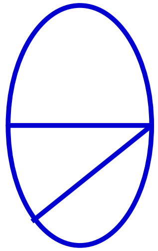

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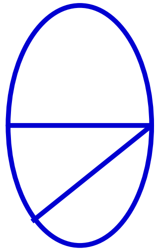

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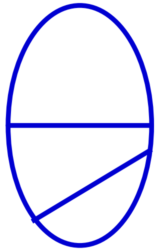

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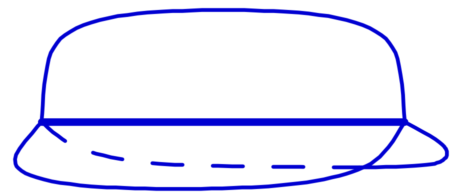
stratified with trivalent 1-strata: 

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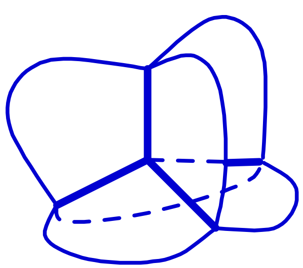
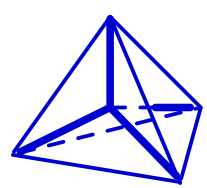
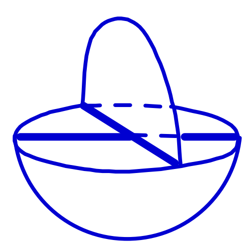
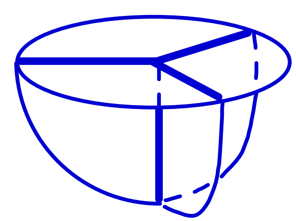
 $\cong$  $\cong$  $\cong$ 

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A generic 2-polyhedron that is not equipped with gleams is considered shadowed with all gleams equal zero.

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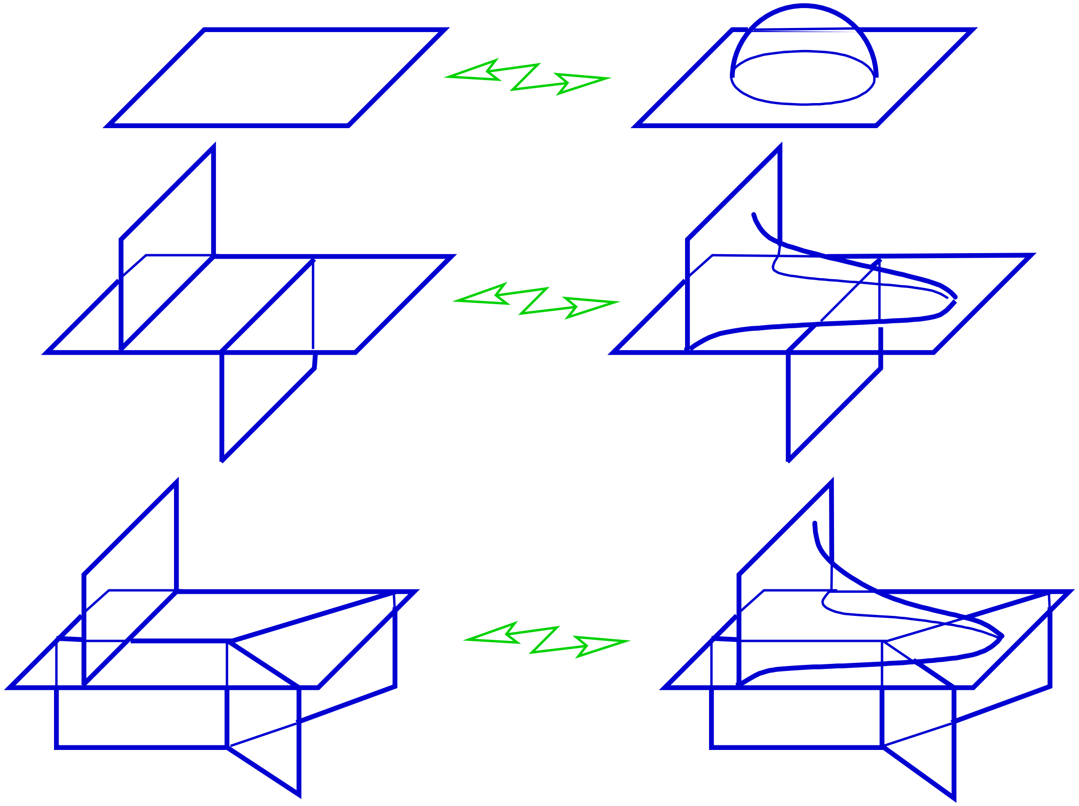

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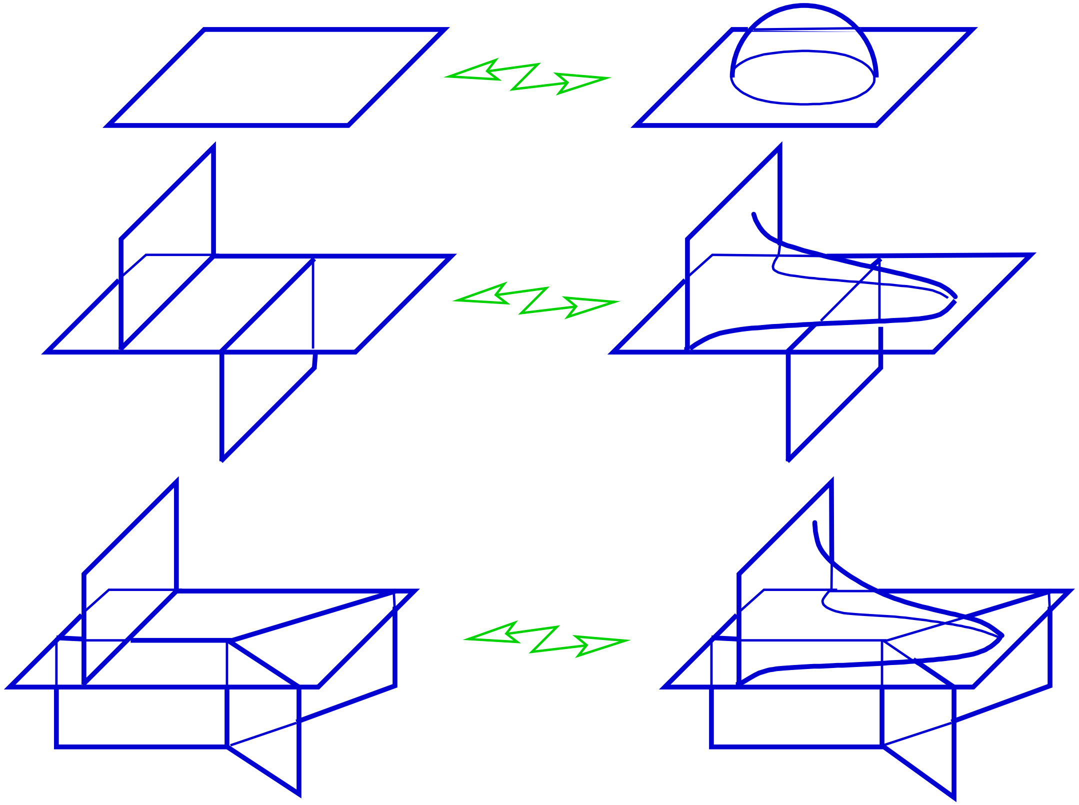
Corollary. Any quantity calculated for a generic 2-polyhedron and invariant with respect the three Matveev-Piergallini moves is a **topological invariant of a 3-manifold**.



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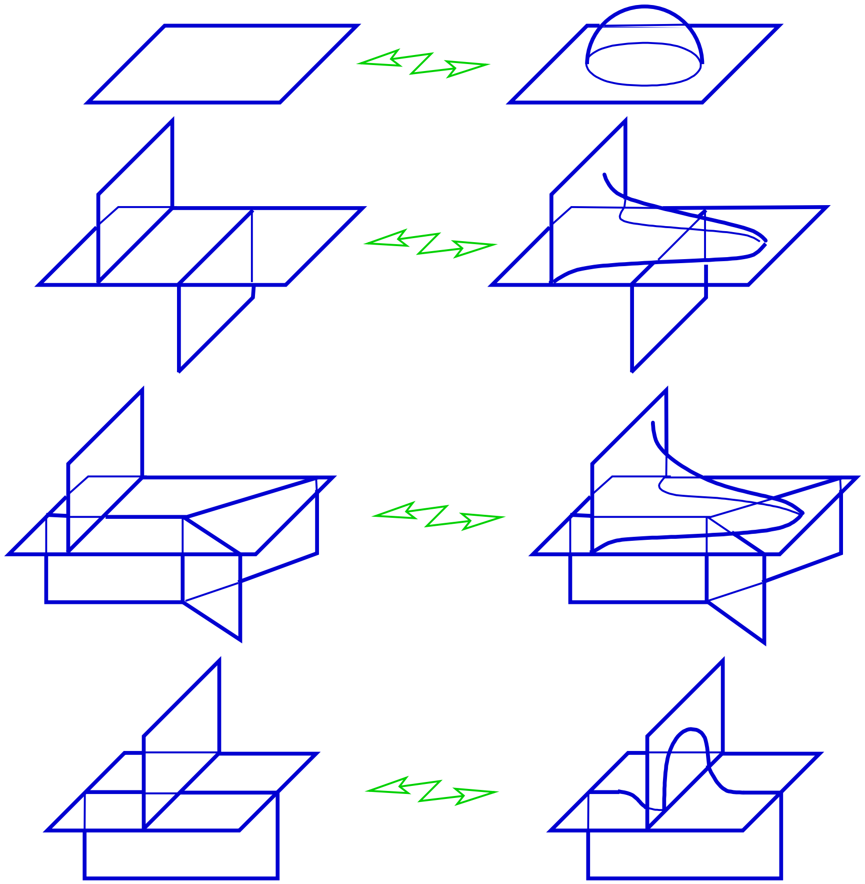

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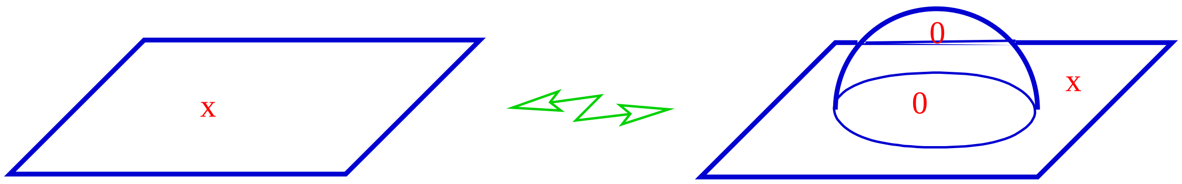

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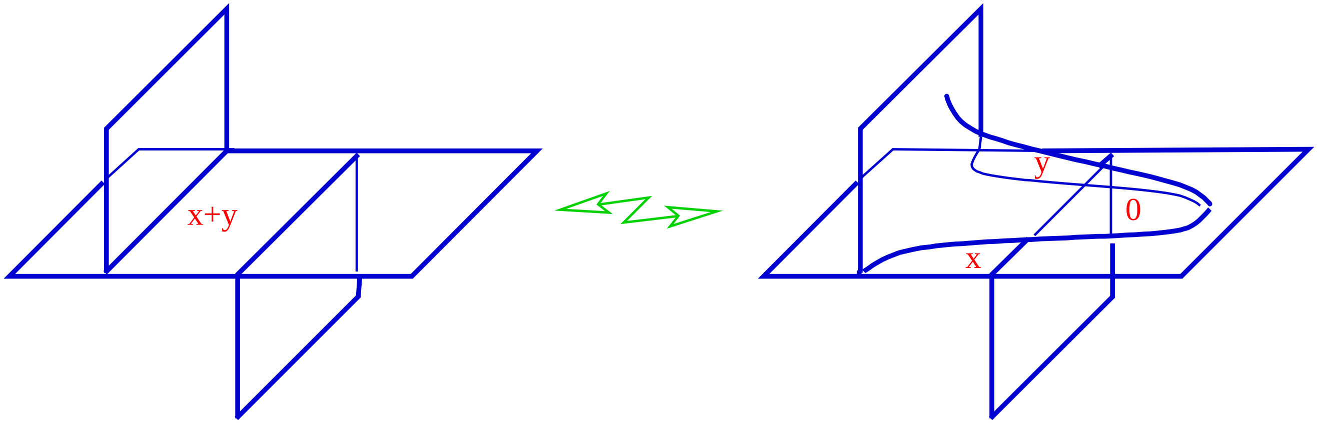

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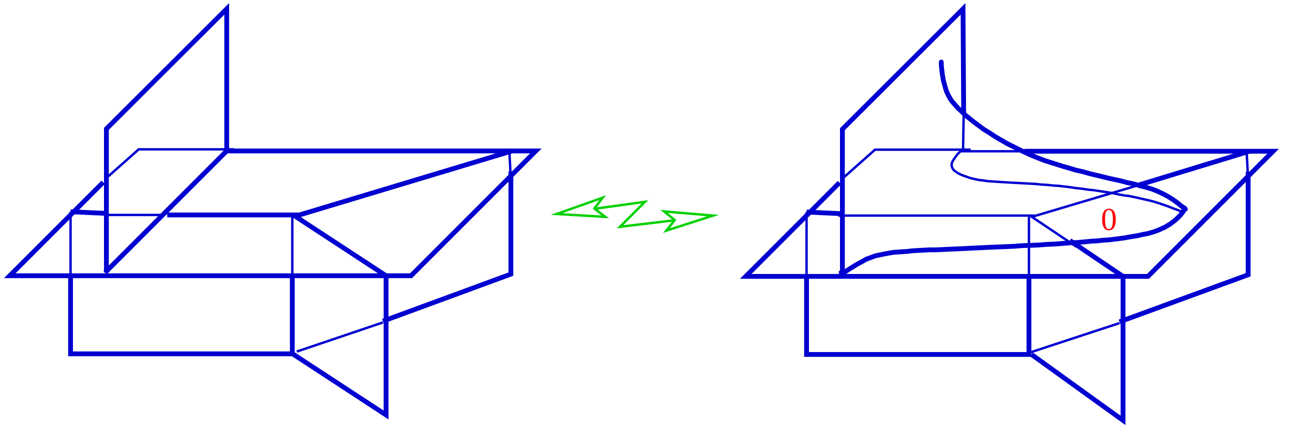

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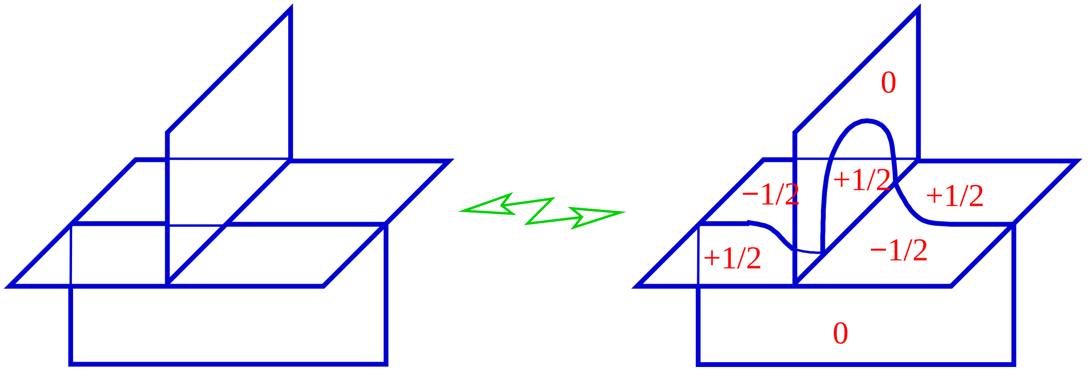
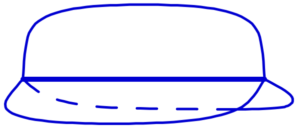
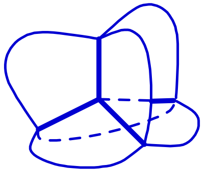
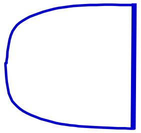
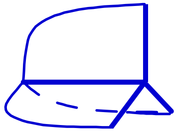


Generic 2-polyhedra with boundary

A generic 2-polyhedron with boundary has interior points with neighborhoods homeomorphic to $\mathbb{R}^2$, or , or , and boundary points with no neighborhoods of these sorts, but with neighborhoods homeomorphic to  or .

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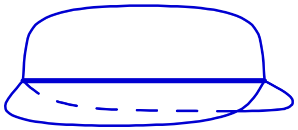
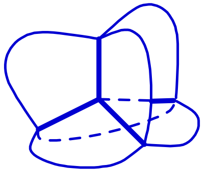
neighborhoods homeomorphic to $\mathbb{R}^2$, or , or ,

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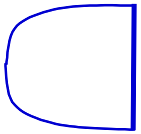
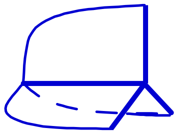
but with neighborhoods homeomorphic to  or .

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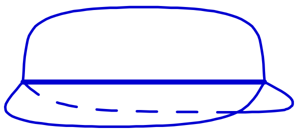
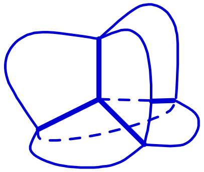
neighborhoods homeomorphic to $\mathbb{R}^2$, or , or ,

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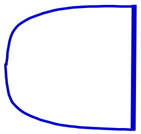
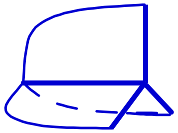
but with neighborhoods homeomorphic to  or .

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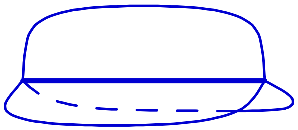
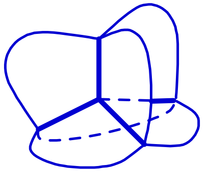
neighborhoods homeomorphic to $\mathbb{R}^2$, or , or ,

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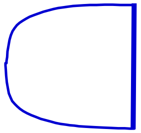
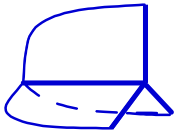
but with neighborhoods homeomorphic to  or .

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Recall: moves do not affect the boundary.

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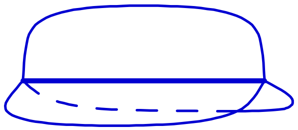
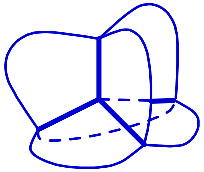
neighborhoods homeomorphic to $\mathbb{R}^2$, or , or ,

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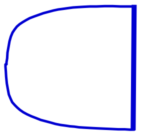
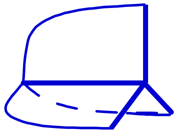
but with neighborhoods homeomorphic to  or .

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Any two trivalent graphs are cobordant, but there are many non-equivalent generic shadowed 2-polyhedra.

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Choose a framing of ∂X in ∂W.

Now all 2-strata of X have self-intersections.

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What w_i and t to choose?

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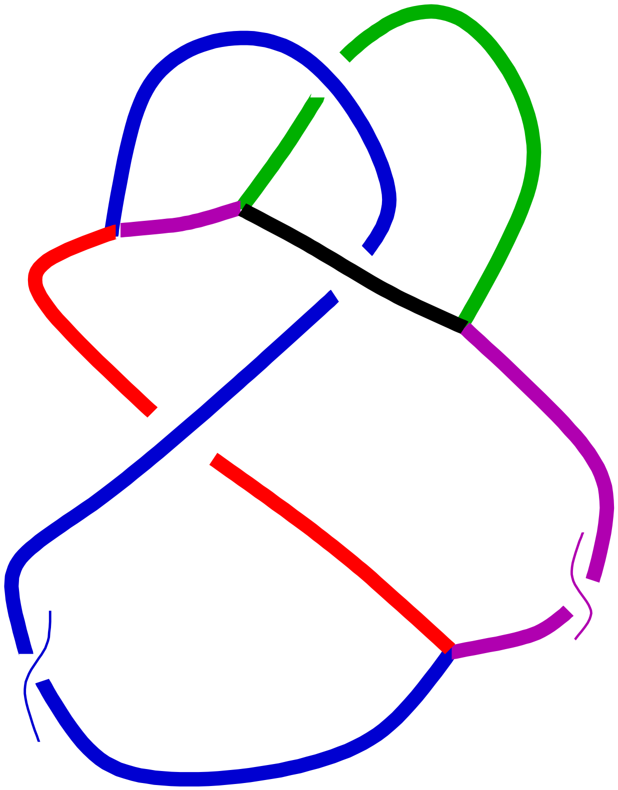

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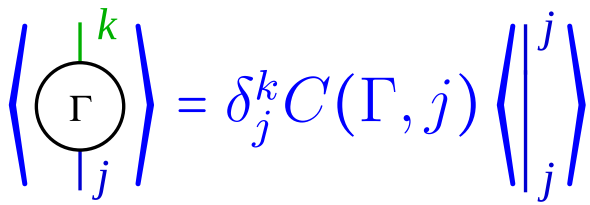

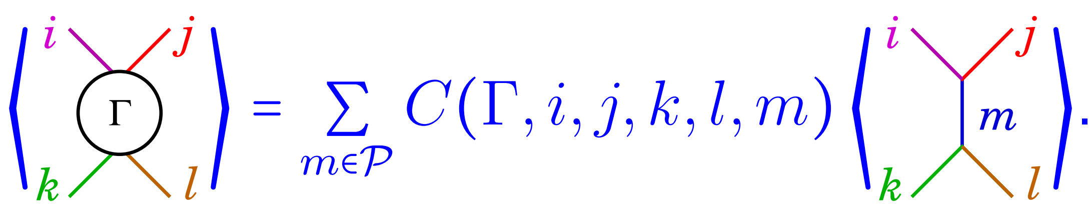

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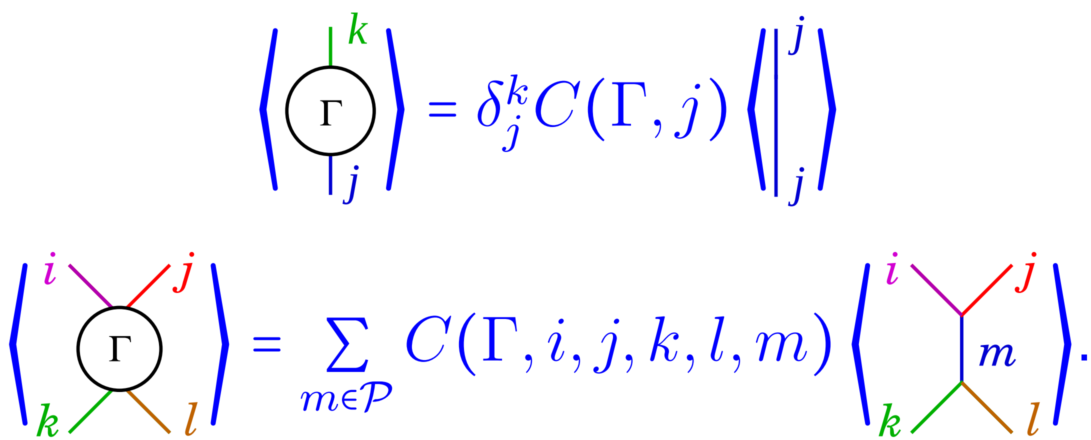
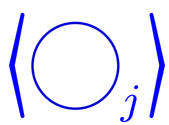
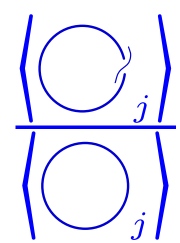
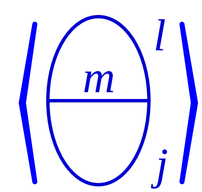
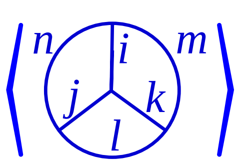


Theorem. If $w_2(j) =$ , $t(j) =$ , $w_1(j, m, l) =$ ,

$w_0\begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix} =$ , $w_3 = \sum_j w_2^2(j)$, then Z_X is invariant under moves and defines a TQFT.

Construction of TQFT

Correction: the state sums define a functor

$$\big(\text{trivalent graphs and their cobordisms}\big) \to \mathrm{Vect}\, k\,.$$

but only a **semifunctor** $\big(\text{manifolds, their cobordisms}\big) \to \mathrm{Vect}\, k\,.$

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In order to turn a functor

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factorize $C(\text{1-skeleton of a manifold} M)$ by $\mathrm{Ker}\, Z_{\text{2-skeleton of } M\times I}$.

Denote $C(\text{1-skeleton of a manifold} M)/\mathrm{Ker}\, Z_{\text{2-skeleton of } M\times I}$ by $Z(M)$ and $Z_{\text{2-skeleton of a cobordism } W}$ by Z_W . **This is a TQFT!**

Old and new TQFT'es

For 2-skeletons of 3-manifolds and the background invariants obtained from the Kauffman bracket extended by cabling and the Jones-Wenzl projectors and evaluated at a root q of unity, this is the Turaev-Viro TQFT introduced in 1991.

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There are many invariants of framed colored trivalent graphs for which the S-matrix is not invertible.

Upgrading the colored Jones

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The only restriction: $H_i \cap S$ is a disk for each i and
in the state sum the colors of these disks
coincide with the colors of the corresponding components of L.

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Let $L = \bigcup_i L_i \subset S^3$ be an oriented classical link framed by its Seifert surface, H_i be a 2-handle attached along L_i and $X = D^4 \cup \bigcup_i H_i$.

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Choose a Seifert surface $F \subset S^3$ for L such that
F is transversal to R and ∂m_i and disjoint from ∂l_i.

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Instead, we will apply it to $S \setminus \cup_i \operatorname{Int} m_i = U$.

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The whole state sum is $J_{L_\lambda}(q) = \sum_\mu \dim_q V_{\mu_1} \ldots \dim_q V_{\mu_n} \operatorname{tr} T_{\lambda,\mu}$.

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If not, how are they related to the Khovanov homology?

Khovanov homology of framed links

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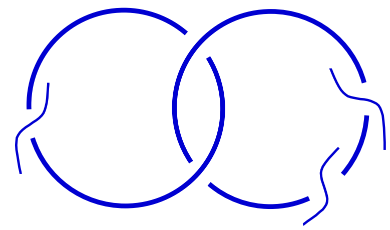
$D =$ 

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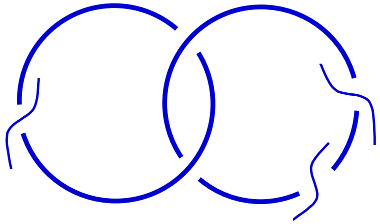
$D =$ 

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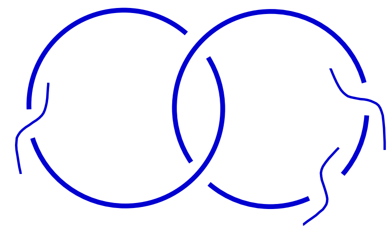
$D =$ 

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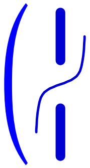
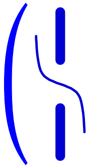
Total framing number $fr(D) = \frac{1}{2}$ ($\#$ () $-$ $\#$ ()).

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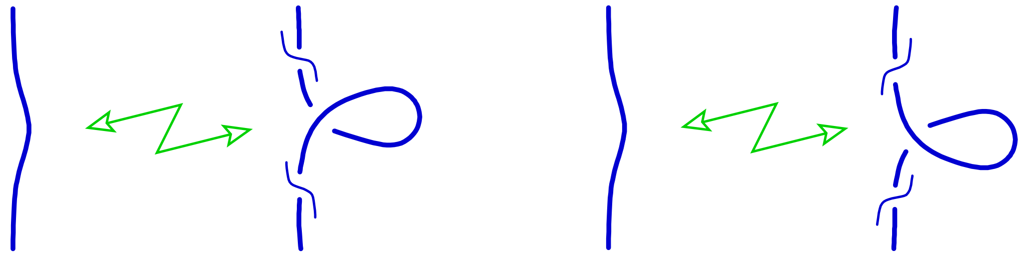
The first Reidemeister move: 

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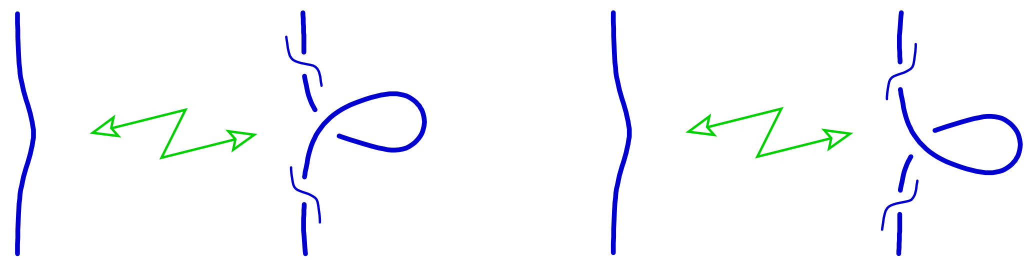
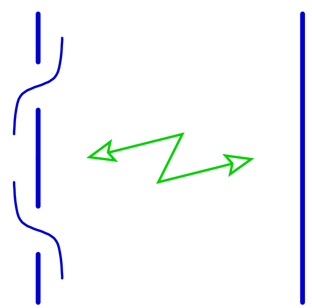
The first Reidemeister move: 

Half-twist annihilation: 

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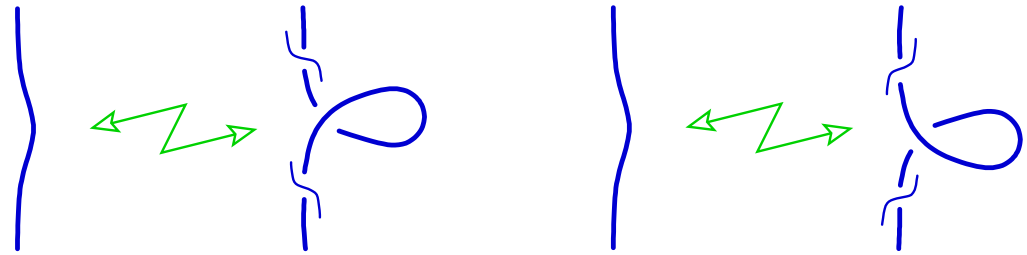
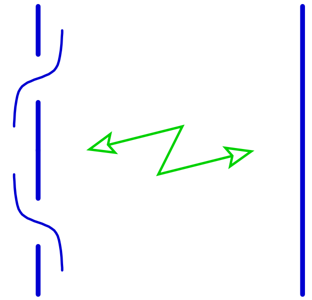
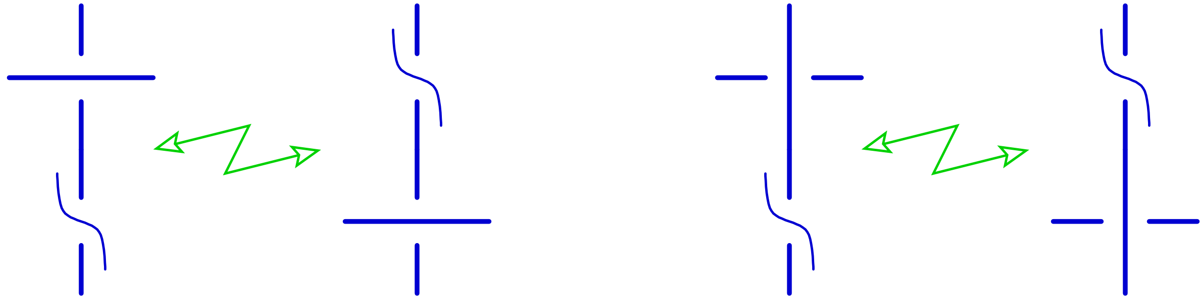
The first Reidemeister move: 

Half-twist annihilation: 

Half-twist penetration: 

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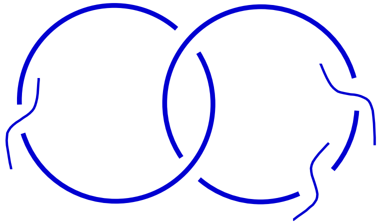
$D =$ 

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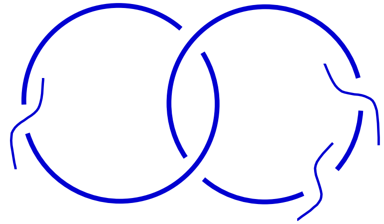
$D =$ 

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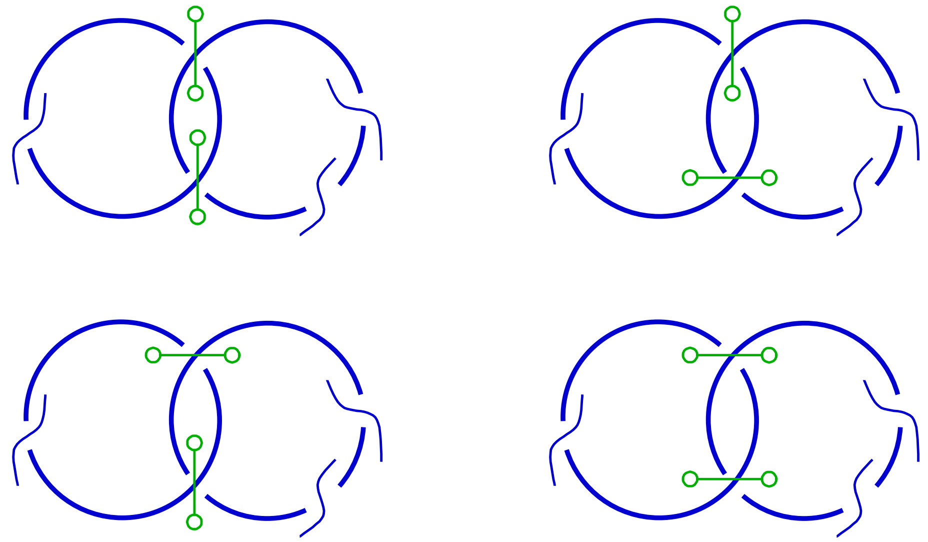

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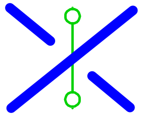
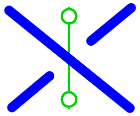
Signs of markers: positive , negative .

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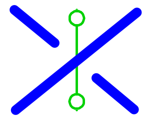
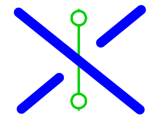
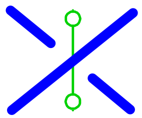
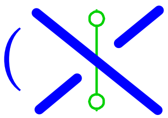
Signs of markers: positive  , negative  .

Numerical characteristics of a Kauffman state s :

$a(s) = \#($  $)$, $b(s) = \#($  $)$, $\sigma(s) = a(s) - b(s)$.

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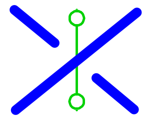
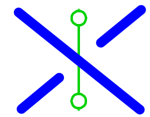
Signs of markers: positive , negative .

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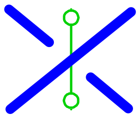
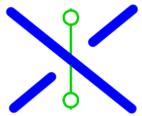
$a(s) = \#($  $)$, $b(s) = \#($  $)$, $\sigma(s) = a(s) - b(s)$.

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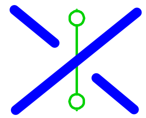
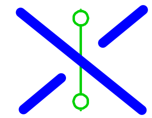
Signs of markers: positive  , negative  .

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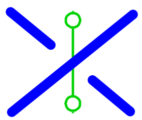
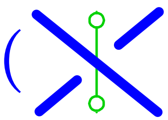
$a(s) = \#($  $)$, $b(s) = \#($  $)$, $\sigma(s) = a(s) - b(s)$.

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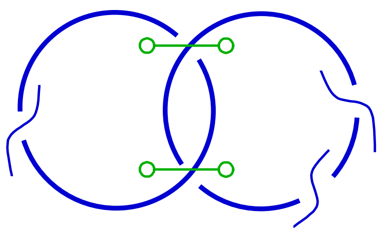
$s =$  $\mapsto$

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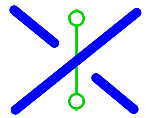
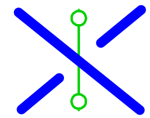
Signs of markers: positive , negative .

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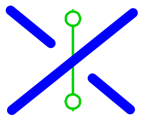
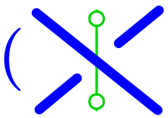
$a(s) = \#($  $)$, $b(s) = \#($  $)$, $\sigma(s) = a(s) - b(s)$.

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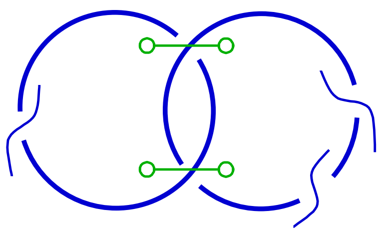
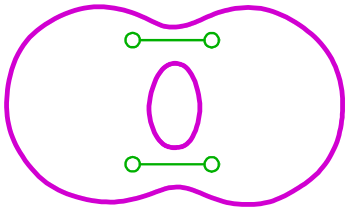
$s =$  $\mapsto D_s =$ 

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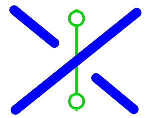
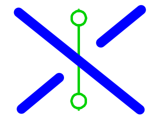
Signs of markers: positive , negative .

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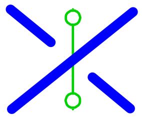
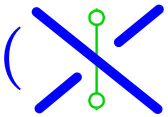
$a(s) = \#($  $)$, $b(s) = \#($  $)$, $\sigma(s) = a(s) - b(s)$.

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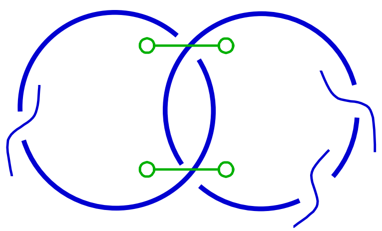
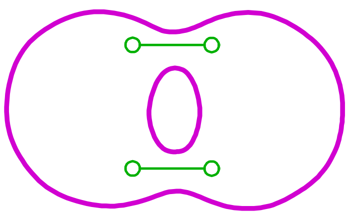
$s =$  $\mapsto D_s =$  $|s| = 2$.

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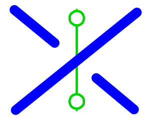
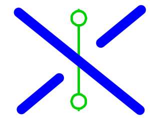
Signs of markers: positive , negative .

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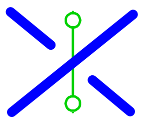
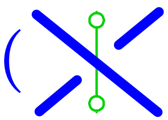
$a(s) = \#($  $)$, $b(s) = \#($  $)$, $\sigma(s) = a(s) - b(s)$.

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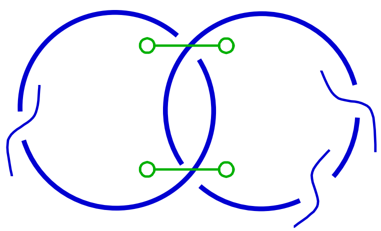
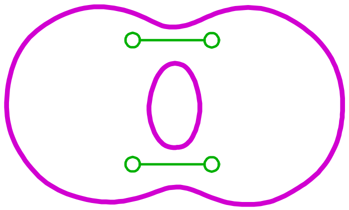
$s =$  $\mapsto D_s =$  $|s| = 2$.

The Kauffman bracket

$$\langle D \rangle = \sum_s (-A)^{3fr(D)} A^{\sigma(s)} (-A^2 - A^{-2})^{|s|}$$

invariant under isotopy of framed links.

Khovanov homology of a framed link

Enhanced state of D is a Kauffman state s of D

plus assignment to every component of D_s either 1 or x.

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Framed Reidemeister moves induce isomorphisms of $Kh^{fr}_{i,j}(D)$.

Cobordisms of framed links

Let D_0 and D_1 be framed links diagrams of L_0 and L_1
and $F \subset \mathbb{R}^3 \times [0,1]$ be a compact surface
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Duality. Let D^* be the mirror image of D. Then the complexes
$C_{i,j}(D^*)$ and $C_{-i,-j}(D)$ are dual, i.e.,
there exists an isomorphism $C_{i,j}(D^*) \to \mathrm{Hom}_{\mathbb{F}_2}(C_{-i,-j}(D))$.

Skein sequences

Kauffman skein relation $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$
categorifies a short exact sequence of complexes:

$$0 \longrightarrow C_{*,*}(\asymp) \xrightarrow{\alpha} C_{*,*-1}(\times) \xrightarrow{\beta} C_{*,*-2}()() \longrightarrow 0\,.$$

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The cylinder of the composition of

$$C_{*,*}(\times) \xrightarrow{\beta} C_{*,*-1}(()()) \xrightarrow{\alpha} C_{*,*-2}(\times)$$

gives rise to a long homology sequence, which
categorifies the Jones skein relation and contains the homomorphism

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Khovanov homology for surfaces in $S^3 \times S^1$

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This can be obtained from a link $\bar{\Lambda} \looparrowright S^4$ by a surgery along an unknotted component of $\bar{\Lambda}$ homeomorphic to S^2.

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Luoying Weng calculated $Z_{i,j}(\Lambda)$ for many such surfaces.

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Theorem. $Z_{i,j}(\Lambda)$ is invariant under isotopy of Λ in $S^3 \times S^1$.

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Because cobordisms needed for Khovanov homology are surfaces in $S^3 \times I$,

while in the proof we meet a cobordism between a link in $S^3 \times \{\mathrm{pt}\}$ and a skew copy of it.

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Pull this new stuff back by $\widetilde{h}_t : S^3 \times \mathbb{R} \to S^3 \times \mathbb{R}$:

$\widetilde{h}_t^{-1}(L_{t,n}) = L_n \subset \widetilde{h}_t^{-1}(S^3 \times \{n\})$,

$\widetilde{h}_t^{-1}(W_{t,n}) = \widetilde{\Lambda} \cap \widetilde{h}_t^{-1}(S^3 \times [n, n+1])$

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