More on Chern-Simons Theory and Feynman Diagrams

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

After $A \mapsto A/\sqrt{k}$, and setting $\hbar = \frac{1}{\sqrt{k}}$:

$$Z(\gamma) = \int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\, tr_R hol_\gamma(A)\, e^{\frac{i}{4\pi}\underbrace{\int_{\mathbb{R}^3} tr(A\wedge dA + \frac{2\hbar}{3}A\wedge A\wedge A)}_{CS(A)}}$$

where $tr_R hol_\gamma(A) = tr_R\Big(1 + \hbar\int ds\, A(\dot\gamma(s))$

$$+\hbar^2\int_{s_1<s_2} A(\dot\gamma(s_1))A(\dot\gamma(s_2)) + \dots$$

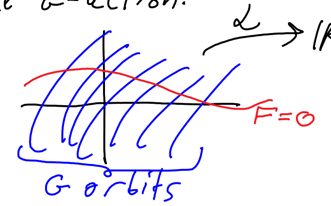
Trouble! "d" is not invertible!

Gauge Invariance: $CS(A)$ is invariant under $A\mapsto A+\delta A$, $\delta A = -(dC + \hbar[A,C])$, $C\in\Omega^0(\mathbb{R},\mathfrak{g})$

Back to the drawing board....

Suppose $\mathcal{L}(x)$ on $\mathbb{R}^n$ is invariant under a k-dimensional group G w/ Lie algebra $\mathfrak{g} = \langle g_a\rangle$, and suppose $F:\mathbb{R}^n\to\mathbb{R}^k$ is such that $F=0$ is a section of the G-action:

$$G \to \mathbb{R}^n \xrightarrow{F} \mathbb{R}^k$$



Then

$$\int_{\mathbb{R}^n} dx\, e^{i\mathcal{L}} \sim \int_{\mathbb{R}^n} dx\, e^{i\mathcal{L}}\delta(F(x))\cdot\det\left(\frac{\partial F^a}{\partial g_b}\right)(x)$$

$$\sim \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^k} d\phi\; e^{i(\mathcal{L}+F(x)\cdot\phi)}\det\left(\frac{\partial F^a}{\partial g_b}\right)(x)$$

perturbation theory for determinants?

$$\det(J_0 + \hbar J_1(x)) = \det(J_0)\sum_m \hbar^m\, Tr\left((\Lambda^m J_0^{-1})\cdot(\Lambda^m J_1(x))\right)$$

Berezin Fermionic Anti-commuting Variables: $\int d^k\bar c\, d^k c\, e^{i\bar c_a J^a_b c^b} \sim \det(J)$

So

$$Z \sim \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^k} d\phi \int d^k\bar c \int d^k c\; e^{i\mathcal{L}_{tot}} \text{ where}$$

$$\mathcal{L}_{tot} = \underbrace{\mathcal{L}(x)}_{\text{the original}} + \underbrace{F(x)\cdot\phi}_{\text{gauge-fixing}} + \underbrace{\bar c_a\left(\frac{\partial F^a}{\partial g_b}\right)c^b}_{\text{"ghosts"}}$$

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org The Knot Atlas

In Chern-Simons, w/ $F(A) := d^*A = \partial_i A^i$, get

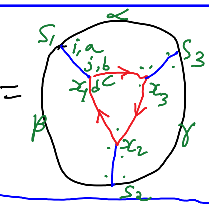
$$\mathcal{L}_{tot} = \frac{k}{4\pi}\int_{\mathbb{R}^3} tr\Big(A\wedge dA + \frac{2}{3}A\wedge A\wedge A + \phi\,\partial_i A^i$$
$$+\bar c\,\partial_i(\partial^i + ad A^i)c$$

So we have

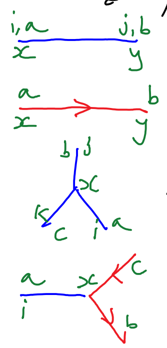
* A bosonic quadratic term involving $\binom{A}{\phi}$.
* A fermionic quadratic term involving $\bar c, c$.
* A cubic interaction of 3 A's.
* A cubic $A\bar c c$ vertex.
* Funny A and γ "holonomy" vertices along γ.

After much crunching:

$$Z(\gamma) = \sum_{m=0}^\infty \hbar^m \sum_{\substack{\text{Feynman}\\ \text{diags } D}} \oint \mathcal{E}(D)$$

$D =$ 

where $\mathcal{E}(D)$ is constructed as follows:

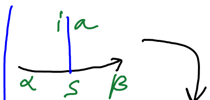


$$\longrightarrow \epsilon_{ijk}t^{ab}\frac{i(x-y)^k}{2|x-y|^3}$$

$$\longrightarrow \frac{t^{ab}}{4\pi|x-y|}$$

$$\longrightarrow \frac{i}{2\pi}\int_{\mathbb{R}^3} t_{abc}\epsilon^{ijk}$$

$$\longrightarrow \frac{1}{2\pi}\int_{\mathbb{R}^3} dz\, t_{abc}\underbrace{\partial^i_x}_{\text{acting only in } b\text{-direction}}$$



$$\int_{S^1} ds\, R^\alpha_{a\beta}\dot\gamma^i(s)$$

$(-)$ sign for each red loop.

By a bit of a miracle, this boils down to a configuration space integral, which in itself, can be reduced to a pre-image count.

... But I run out of steam for tonight...

Banks like knots.