Meta−Groups, Meta−Bicrossed−Products, and the Alexander Polynomial, 1

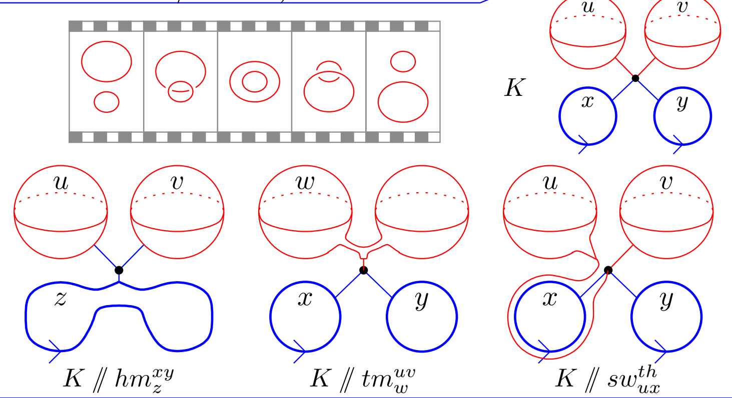
Dror Bar−Natan at the Newton Institute, January 2013.

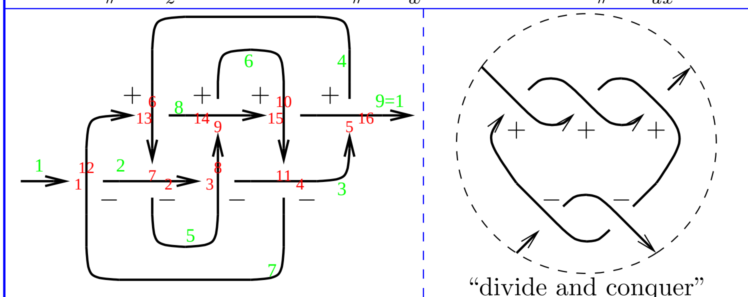
http://www.math.toronto.edu/~drorbn/Talks/Newton-1301

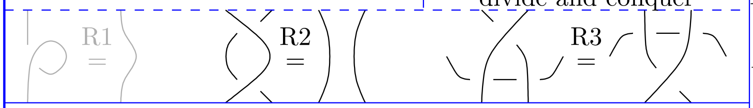
Abstract. I will define "meta-groups" and explain how one specific meta-group, which in itself is a "meta-bicrossed-product", gives rise to an "ultimate Alexander invariant" of tangles, that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, that's a wonderful playground.

This will be a repeat of a talk I gave in Regina in August 2012 and in a number of other places, and I plan to repeat it a good further number of places. Though here at the Newton Institute I plan to make the talk a bit longer, giving me more time to give some further fun examples of meta-structures, and perhaps I will learn from the audience that these meta-structures should really be called something else.

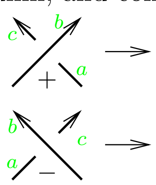
This work is closely related to work by Le Dimet (Comment. Math. Helv. **67** (1992) 306–315), Kirk, Livingston and Wang (arXiv:math/9806035) and Cimasoni and Turaev (arXiv:math.GT/0406269).







A Standard Alexander Formula. Label the arcs 1 through $(n+1)=1$, make an $n\times n$ matrix as below, delete one row and one column, and compute the determinant:



	a	b	c
c	-1	$1-X$	X

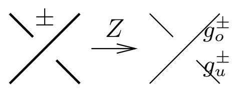
	a	b	c
c	$-X$	$X-1$	1

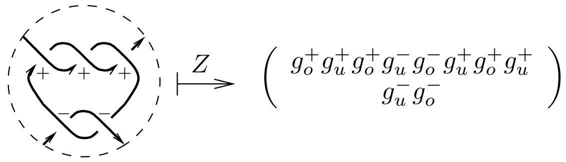
```
( 1     0     0     0     0     X - 1  0      -X    )
( -1    X     0     0     0     0      1 - X  0     )
( 0     -1    X     0     1 - X 0      0      0     )
( X - 1 0     -X    1     0     0      0      0     )  [[1 ;; 7, 1 ;; 7]] // Det
( 0     1 - X 0     -1    X     0      0      0     )
( 0     0     0     0     -X    1      0      X - 1 )
( 0     0     1 - X 0     0     -1     X      0     )
( 0     0     0     X - 1 0     0      -X     1     )

-1 + 4 X - 8 X^2 + 11 X^3 - 8 X^4 + 4 X^5 - X^6
```

Alexander Issues.

- Quick to compute, but computation departs from topology.
- Extends to tangles, but at an exponential cost.
- Hard to categorify.

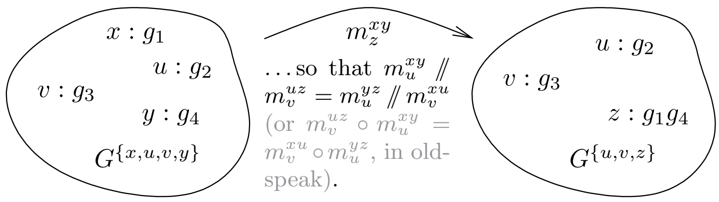
Idea. Given a group G and two "YB" pairs $R^\pm=(g^\pm_o,g^\pm_u)\in G^2$, map them to xings and "multiply along", so that





This Fails! R2 implies that $g^\pm_o g^\mp_o = e = g^\pm_u g^\mp_u$ and then R3 implies that g^+_o and g^+_u commute, so the result is a simple counting invariant.

A Group Computer. Given G, can store group elements and perform operations on them:



Also has S_x for inversion, e_x for unit insertion, d_x for register deletion, Δ^z_{xy} for element cloning, ρ^x_y for renamings, and $(D_1,D_2)\mapsto D_1\cup D_2$ for merging, and many obvious composition axioms relating those. $P=\{x:g_1,y:g_2\}\Rightarrow P=\{d_yP\}\cup\{d_xP\}$

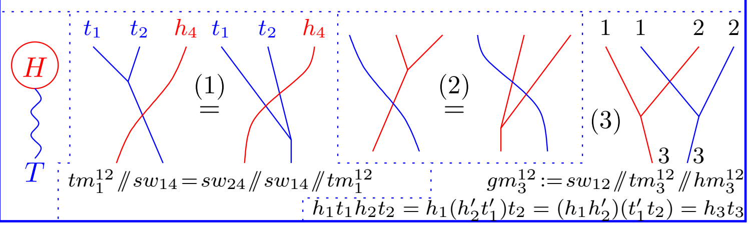
A Meta-Group. Is a similar "computer", only its internal structure is unknown to us. Namely it is a collection of sets $\{G_\gamma\}$ indexed by all finite sets γ, and a collection of operations m^{xy}_z, S_x, e_x, d_x, Δ^z_{xy} (sometimes), ρ^x_y, and $\cup$, satisfying the exact same *linear* properties.

Example 1. The non-meta example, $G_\gamma := G^\gamma$.

Example 2. $G_\gamma := M_{\gamma\times\gamma}(\mathbb{Z})$, with simultaneous row and column operations, and "block diagonal" merges. Here if $P=\begin{pmatrix} x: & a & b\\ y: & c & d\end{pmatrix}$ then $d_yP=(x:a)$ and $d_xP=(y:d)$ so $\{d_yP\}\cup\{d_xP\}=\begin{pmatrix} x: & a & 0\\ y: & 0 & d\end{pmatrix}\neq P$. So this G is truly meta.

Claim. From a meta-group G and YB elements $R^\pm\in G_2$ we can construct a knot/tangle invariant.

Bicrossed Products. If $G=HT$ is a group presented as a product of two of its subgroups, with $H\cap T=\{e\}$, then also $G=TH$ and G is determined by H, T, and the "swap" map $sw^{th}:(t,h)\mapsto(h',t')$ defined by $th=h't'$. The map sw satisfies (1) and (2) below; conversely, if $sw:T\times H\to H\times T$ satisfies (1) and (2) (+ lesser conditions), then (3) defines a group structure on $H\times T$, the "bicrossed product".



Meta−Groups, Meta−Bicrossed−Products, and the Alexander Polynomial, 2

A **Meta-Bicrossed-Product** is a collection of sets $\beta(\eta,\tau)$ and operations tm_z^{xy}, hm_z^{xy} and sw_{xy}^{th} (and lesser ones), such that tm and hm are "associative" and (1) and (2) hold (+ lesser conditions). A meta-bicrossed-product defines a meta-group with $G_\gamma := \beta(\gamma,\gamma)$ and gm as in (3).

Example. Take $\beta(\eta,\tau) = M_{\tau\times\eta}(\mathbb{Z})$ with row operations for the tails, column operations for the heads, and a trivial swap.

β Calculus. Let $\beta(\eta,\tau)$ be

$$\left\{\begin{array}{c|cccc} \omega & h_1 & h_2 & \cdots \\ \hline t_1 & \alpha_{11} & \alpha_{12} & \cdot \\ t_2 & \alpha_{21} & \alpha_{22} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \;\middle|\; \text{$h_j\in\eta$, $t_i\in\tau$, and ω and the α_{ij} are rational functions in a variable X}\right\},$$

$$tm_z^{xy}: \begin{array}{c|c} \omega & \cdots \\ \hline t_x & \alpha \\ t_y & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline t_z & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & \eta_1 \\ \hline \tau_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & \eta_2 \\ \hline \tau_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & \eta_1 & \eta_2 \\ \hline \tau_1 & \alpha_1 & 0 \\ \tau_2 & 0 & \alpha_2 \end{array},$$

$$hm_z^{xy}: \begin{array}{c|cccc} \omega & h_x & h_y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & h_z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

$$sw_{xy}^{th}: \begin{array}{c|cc} \omega & h_y & \cdots \\ \hline t_x & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & h_y & \cdots \\ \hline t_x & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1+\alpha$ and $\langle c\rangle := \sum_i c_i$, and let

$$R_{xy}^p := \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & X-1 \\ t_y & 0 & 0 \end{array} \qquad R_{xy}^m := \begin{array}{c|cc} 1 & h_x & h_y \\ \hline t_x & 0 & X^{-1}-1 \\ t_y & 0 & 0 \end{array}.$$

Theorem. Z^β is a tangle invariant (and more). Restricted to knots, the ω part is the Alexander polynomial. On braids, it is equivalent to the Burau representation. A variant for links contains the multivariable Alexander polynomial.

Why Happy? • Applications to w-knots.
• Everything that I know about the Alexander polynomial can be expressed cleanly in this language (even if without proof), except HF, but including genus, ribbonness, cabling, v-knots, knotted graphs, etc., and there's potential for vast generalizations.
• The least wasteful "Alexander for tangles" I'm aware of.
• Every step along the computation is the invariant of something.
• Fits on one sheet, including implementation & propaganda.

Further meta-monoids. Π (and variants), $\mathcal{A}$ (and quotients), vT, …

Further meta-bicrossed-products. Π (and variants), $\overrightarrow{\mathcal{A}}$ (and quotients), M_0, M, $\mathcal{K}^{bh}$, $\mathcal{K}^{rbh}$, …

Meta-Lie-algebras. $\mathcal{A}$ (and quotients), $\mathcal{S}$, …

Meta-Lie-bialgebras. $\overrightarrow{\mathcal{A}}$ (and quotients), …

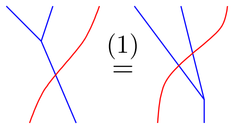
I mean business!

```
βSimp = Factor; SetAttributes[βCollect, Listable];
βCollect[B[ω_, Λ_]] := B[βSimp[ω],
   Collect[Λ, h_, Collect[#, t_, βSimp] &]];
βForm[B[ω_, Λ_]] := Module[{ts, hs, M},
   ts = Union[Cases[B[ω, Λ], t_s_ :> s, Infinity]];
   hs = Union[Cases[B[ω, Λ], h_s_ :> s, Infinity]];
   M = Outer[βSimp[Coefficient[Λ, h_#1 t_#2]] &, hs, ts];
   PrependTo[M, t_# & /@ ts];
   M = Prepend[Transpose[M], Prepend[h_# & /@ hs, ω]];
   MatrixForm[M]];
βForm[else_] := else /. β_B :> βForm[β];
Format[β_B, StandardForm] := βForm[β];
```

```
⟨μ_⟩ := μ /. t_ → 1;
tm_{x_ y_→z_}[β_] := βCollect[β /. t_{x|y} → t_z];
hm_{x_ y_→z_}[B[ω_, Λ_]] := Module[
   {α = D[Λ, h_x], β = D[Λ, h_y], γ = Λ /. h_{x|y} → 0},
   B[ω, (α + (1 + ⟨α⟩) β) h_z + γ] // βCollect];
sw_{x_ y_}[B[ω_, Λ_]] := Module[{α, β, γ, δ, ϵ},
   α = Coefficient[Λ, h_y t_x]; β = D[Λ, t_x] /. h_y → 0;
   γ = D[Λ, h_y] /. t_x → 0;    δ = Λ /. h_y | t_x → 0;
   ϵ = 1 + α;
   B[ω * ϵ, α (1 + ⟨γ⟩ / ϵ) h_y t_x + β (1 + ⟨γ⟩ / ϵ) t_x
        + γ / ϵ h_y                 + δ - γ * β / ϵ
    ] // βCollect];
gm_{x_ y_→z_}[β_] := β // sw_{xy} // hm_{xy→z} // tm_{xy→z};
B /: B[ω1_, Λ1_] B[ω2_, Λ2_] := B[ω1 * ω2, Λ1 + Λ2];
Rp_{x_ y_} := B[1, (X - 1) t_x h_y];
Rm_{x_ y_} := B[1, (X^-1 - 1) t_x h_y];
```

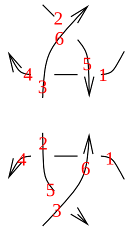
```
{β = B[ω, Sum[α_{10 i+j} t_i h_j, {i, {1, 2, 3}}, {j, {4, 5}}]],
 (β // tm_{12→1} // sw_{14}) == (β // sw_{24} // sw_{14} // tm_{12→1})}
```

$$\left\{\begin{pmatrix} \omega & h_4 & h_5 \\ t_1 & \alpha_{14} & \alpha_{15} \\ t_2 & \alpha_{24} & \alpha_{25} \\ t_3 & \alpha_{34} & \alpha_{35} \end{pmatrix}, \text{True}\right\}$$



Some testing

```
{Rm_{51} Rm_{62} Rp_{34} // gm_{14→1} // gm_{25→2} // gm_{36→3},
 Rp_{61} Rm_{24} Rm_{35} // gm_{14→1} // gm_{25→2} // gm_{36→3}}
```



$$\left\{\begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+X}{X} & 0 \\ t_3 & \frac{-1+X}{X} & -\frac{-1+X}{X} \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+X}{X} & 0 \\ t_3 & \frac{-1+X}{X} & -\frac{-1+X}{X} \end{pmatrix}\right\}$$

… divide and conquer!

```
β = Rm_{12,1} Rm_{27} Rm_{83} Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15}
```

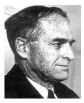
8_{17}

1	h_1	h_3	h_5	h_7	h_9	h_{11}	h_{13}	h_{15}
t_2	0	0	0	$-\frac{-1+X}{X}$	0	0	0	0
t_4	0	0	0	0	0	$-\frac{-1+X}{X}$	0	0
t_6	0	0	0	0	0	0	$-1+X$	0
t_8	0	$-\frac{-1+X}{X}$	0	0	0	0	0	0
t_{10}	0	0	0	0	0	0	0	$-1+X$
t_{12}	$-\frac{-1+X}{X}$	0	0	0	0	0	0	0
t_{14}	0	0	0	0	$-1+X$	0	0	0
t_{16}	0	0	$-1+X$	0	0	0	0	0

```
Do[β = β // gm_{1k→1}, {k, 2, 10}]; β
```

8_{17}, cont.

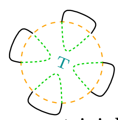
$\frac{1}{X}$	h_1	h_{11}	h_{13}	h_{15}
t_1	$-\frac{(-1+X)(1+X)}{X}$	$-(-1+X)(1-X+X^2)$	$(-1+X)(1-X+X^2)$	$-1+X$
t_{12}	$-\frac{-1+X}{X}$	0	0	0
t_{14}	$-1+X$	$\frac{(-1+X)^2(1-X+X^2)}{X}$	$-\frac{(-1+X)^2(1-X+X^2)}{X}$	0
t_{16}	$\frac{-1+X}{X}$	$(-1+X)^2$	$-\frac{(-1+X)^3}{X}$	0

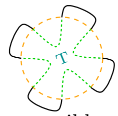

James Waddell Alexander

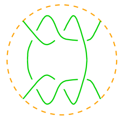
```
Do[β = β // gm_{1k→1}, {k, 11, 16}]; β
```

$$\left(-\frac{1-4X+8X^2-11X^3+8X^4-4X^5+X^6}{X^3}\right)$$

A Partial To Do List. 1. Where does it *more simply* come from?
2. Remove all the denominators.
3. How do determinants arise in this context?
4. Understand links.
5. Find the "reality condition".
6. Do some "Algebraic Knot Theory".
7. Categorify.
8. Do the same in other natural quotients of the v/w-story.


trivial


ribbon


example

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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