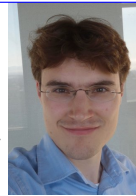




Knot Invariants from Finite Dimensional Integration

web:=http://drorbn.net/g24

Abstract. For the purpose of today, an “I-Type Knot Invariant” is a knot invariant computed from a knot diagram by integrating the exponential of a *perturbed Gaussian* Lagrangian which is a sum over the features of that diagram (crossings, edges, faces) of locally defined quantities, over a product of finite dimensional spaces associated to those same features.



joint with R. van der Veen



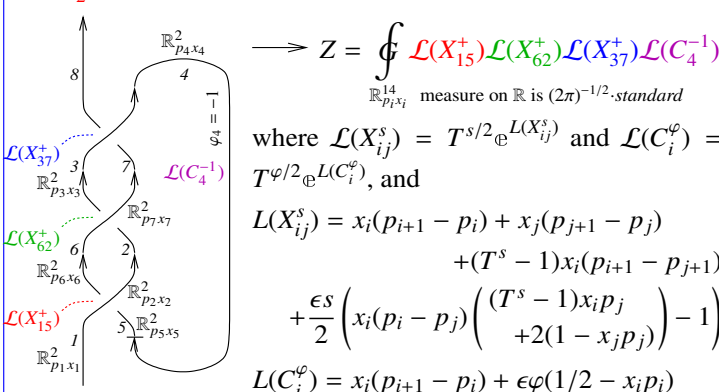
Knots.

Something simple: numbers, polynomials, matrices, etc.

- Q. Are there any such things? **A.** Yes.
- Q. Are they any good? **A.** They are the strongest we know per CPU cycle, and are excellent in other ways too.
- Q. Didn't Witten do that back in 1988 with path integrals? **A.** No. His constructions are infinite dimensional and far from rigorous.
- Q. But integrals belong in analysis! **A.** Ours only use squeaky-clean algebra.



The s/ϵ^2 Example. With T an indeterminate and with $\epsilon^2 = 0$:

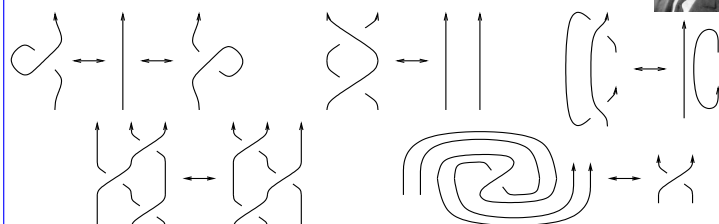


So $Z = T \int e^{L(\odot)} dp_1 \dots dp_7 dx_1 \dots dx_7$, where $L(\odot) =$

$$\sum_{i=1}^7 x_i(p_{i+1} - p_i) + (T-1)(x_1(p_2 - p_6) + x_6(p_7 - p_3) + x_3(p_4 - p_8)) + \frac{\epsilon}{2} \begin{pmatrix} x_1(p_1 - p_5) ((T-1)x_1 p_5 + 2(1 - x_5 p_5)) - 1 \\ +x_6(p_6 - p_2) ((T-1)x_6 p_2 + 2(1 - x_2 p_2)) - 1 \\ +x_3(p_3 - p_7) ((T-1)x_3 p_7 + 2(1 - x_7 p_7)) - 1 \\ +2x_4 p_4 - 1 \end{pmatrix}$$

and so $Z = (T - 1 + T^{-1})^{-1} \exp\left(\epsilon \cdot \frac{(T-2+T^{-1})(T+T^{-1})}{(T-1+T^{-1})^2}\right) = \Delta^{-1} \exp\left(\epsilon \cdot \frac{(T-2+T^{-1})\rho_1}{\Delta^{-2}}\right)$. Here Δ is Alexander's polynomial and ρ_1 is Rozansky-Overbay's polynomial [R1]–[R3], [Ov, BV1, BV2].

Theorem. Z is a knot invariant.
Proof. Use Fubini (details later).



- To Do.** • Human-hard but computer-very-easy (poly time!).
- Strong! • Details of the proof. • Where is it coming from?
- A philosophical point: “Universal invariants” are qualitatively better than representation theory ones.

The Good. 1. At the centre of low dimensional topology.
2. “Invariants” connect to pretty much all of algebra.

The Agony. 1&2 don't talk to each other.

- Not enough topological applications for all these invariants.
- The fancy algebra doesn't arise naturally within topology.

⇒ We're still missing something about the relationship between knots and algebra.

(Alternative) Gaussian Integration.



Goal. Compute $\int_{\mathbb{R}^n} dx \exp\left(-\frac{1}{2} a^{ij} x_i x_j + V(x)\right)$ (if convergent)

Solution. Set $\mathcal{Z}_\lambda(x) := \lambda^{n/2} \int_{\mathbb{R}^n} dy \exp\left(-\frac{1}{2\lambda} a^{ij} y_i y_j + V(x+y)\right)$. Then $\mathcal{Z}_1(0)$ is what we want, $\mathcal{Z}_0(x) = (\det A)^{-1/2} \exp V(x)$, and with g_{ij} the inverse matrix of a^{ij} and noting that under the dy integral $\partial_y = 0$,

$$\begin{aligned} &= \frac{1}{2} \int_{\mathbb{R}^n} dy g_{ij} (\partial_{x_i} - \partial_{y_i}) (\partial_{x_j} - \partial_{y_j}) \exp\left(-\frac{1}{2\lambda} a^{ij} y_i y_j + V(x+y)\right) \\ &= \frac{1}{2\lambda^2} \int_{\mathbb{R}^n} dy (g_{ij} a^{ii'} a^{jj'} y_{i'} y_{j'} + \lambda g_{ij} a^{ji}) \exp\left(-\frac{1}{2\lambda} a^{ij} y_i y_j + V(x+y)\right) \\ &= \frac{1}{2\lambda^2} \int_{\mathbb{R}^n} dy (a^{ij} y_i y_j + \lambda n) \exp\left(-\frac{1}{2\lambda} a^{ij} y_i y_j + V(x+y)\right) \\ &= \partial_\lambda \mathcal{Z}_\lambda(x). \end{aligned}$$

Hence $(*) \partial_\lambda \mathcal{Z}_\lambda(x) = \frac{1}{2} g_{ij} \partial_{x_i} \partial_{x_j} \mathcal{Z}_\lambda(x)$, and therefore $\mathcal{Z}_\lambda(x) = (\det A)^{-1/2} \exp\left(\frac{\lambda}{2} g_{ij} \partial_{x_i} \partial_{x_j}\right) \exp V(x)$.

We've just witnessed the birth of “Feynman Diagrams”.



Even better. With $Z_\lambda := \log(\sqrt{\det A} \mathcal{Z}_\lambda)$, by a simple substitution into (*), we get the “Synthesis Equation”:

$$Z_0 = V, \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j=1}^n g_{ij} (\partial_{x_i x_j} Z_\lambda + (\partial_{x_i} Z_\lambda)(\partial_{x_j} Z_\lambda)),$$

an ODE (in λ) whose solution is pure algebra.

Picard Iteration (used to prove the existence and uniqueness of solutions of ODEs). To solve $\partial_\lambda f_\lambda = F(f_\lambda)$ with a given f_0 , start with f_0 , iterate $f \mapsto f_0 + \int_0^1 F(f_\lambda) d\lambda$, and seek a fixed point. In our cases, it is always reached after finitely many iterations!



Definition. \mathcal{f} : The result of this process, ignoring the convergence of the actual integral.

Strong. A faster program to compute ρ_1 is available at [BV2]. With it we find that the pair (Δ, ρ_1) attains 53,684 distinct values on the 59,937 prime knots with up to 14 crossings (a deficit of 6,253), whereas the pair (HOMFLYPT polynomial, Khovanov Homology) attains only 49,149 distinct values on the same knots (a deficit of 10,788).

In as much as we know the pair (Δ, ρ_1) is the strongest knot invariant that can be computed in **polynomial time** (and hence, even for very large knots).