A Seifert Dream

Thanks for inviting me to Pitzer College!

Abstract. Given a knot K with a Seifert surface Σ, I dream that the well-known Seifert linking form Q, a quadratic form on $H_1(\Sigma)$, has plenty docile local perturbations P_ϵ such that the formal Gaussian integrals of $\exp(Q+P_\epsilon)$ are invariants of K.
In my talk I will explain what the above means, why this dream is oh so sweet, and why it is in fact closer to a plan than to a delusion. Joint with Roland van der Veen.

The Seifert-Alexander Formula. With $P, Q \in H_1(\Sigma)$,

$$Q(P,G) = T^{1/2} lk(P^+, G) - T^{-1/2} lk(P, G^+)$$

$$\Delta(K) = \det(Q)$$

$$\int_{2H_1(\Sigma)} dp\,dx\,\exp Q(p,x) \doteq \det(Q)^{-1}$$

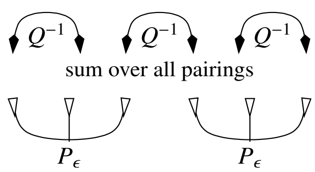
(where $\doteq$ means "ignoring silly factors").

From Mexico City, tariffs exempt

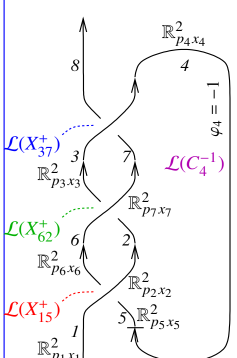
Perturbed Gaussian Integration. We say that $P_\epsilon \in \epsilon\mathbb{Q}[x_1,\dots x_n][\![\epsilon]\!]$ is M-docile (for some $M\colon \mathbb{N}\to\mathbb{N}$) if for every monomial m in P_ϵ we have $\deg_{x_1,\dots,x_n}(m) \leq M(\deg_\epsilon(m))$.

Theorem (Feynman). If Q is a quadratic in $x_1,\dots,x_n$ and P_ϵ is docile, set $Z_\epsilon = \int_{\mathbb{R}^n} dx_1\cdots x_n \exp(Q+P_\epsilon)$. Then every coefficient in the ϵ-expansion of Z_ϵ is computable in polynomial time in n. in fact,

$$\Delta^{1/2} Z_\epsilon \doteq \left\langle \exp Q^{-1}(\partial_{x_i}), \exp P_\epsilon \right\rangle =$$



$\theta(T,1)$ is like that! With $\epsilon^2=0$,



$$\longrightarrow Z \doteq \oint_{2E=\mathbb{R}^{14}_{p_ix_i}} \mathcal{L}(X^+_{15})\mathcal{L}(X^+_{62})\mathcal{L}(X^+_{37})\mathcal{L}(C_4^{-1})$$

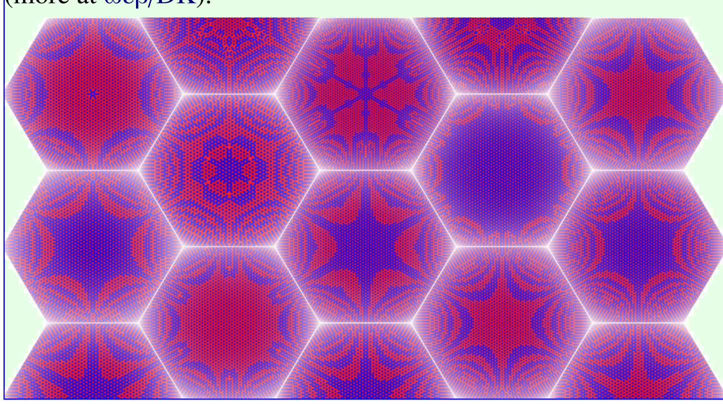
where $\mathcal{L}(X^s_{ij}) \doteq \mathbb{e}^{L(X^s_{ij})}$, $\mathcal{L}(C^\varphi_i) \doteq \mathbb{e}^{L(C^\varphi_i)}$,

$$L(X^s_{ij}) = x_i(p_{i+1}-p_i) + x_j(p_{j+1}-p_j) + (T^s-1)x_i(p_{i+1}-p_{j+1}) + \frac{\epsilon s}{2}\left(x_i(p_i-p_j)\begin{pmatrix}(T^s-1)x_ip_j\\ +2(1-x_jp_j)\end{pmatrix}-1\right)$$

$$L(C^\varphi_i) = x_i(p_{i+1}-p_i) + \epsilon\varphi(1/2 - x_ip_i)$$

$\theta(T_1,T_2)$ is likewise, with harder formulas and integration over $6E$.

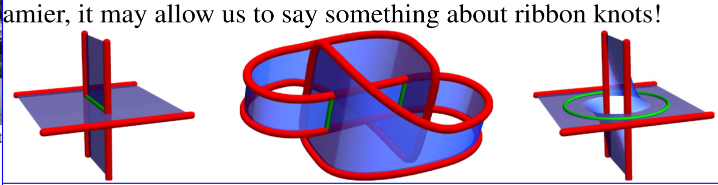
Right. The 132-crossing torus knot $T_{22/7}$ (more at ωεβ/TK).
Below. Random knots from [DHOEBL], with 101-115 crossings (more at ωεβ/DK).



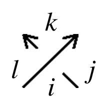
Dream. There is a similar perturbed Gaussian integral formula for θ, but with integration over $6H_1(\Sigma)$. The quadratic Q will be the same as in the Seifert-Alexander formula (but repeated 3 times, for each T_ν). The perturbation P_ϵ will be given by low-degree finite type invariants of curves on Σ (possibly also dependent on the intersection points of such curves, or on other information coming from Σ).

Evidence. Experimentally (yet undeniably), $\deg\theta$ is bounded by the genus of Σ. How else could such a genus bound arise? Further very strong evidence comes from the conjectural (yet undeniable) understanding of θ as the two-loop contribution to the Kontsevich integral [Oh] and/or as the "solvable approximation" of the universal sl_3 invariant [BN1, BV2].

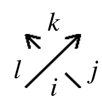
Why so sweet? It will allow us to prove the aforementioned genus bound and likely, the hexagonal symmetry. Sweeter and dreamier, it may allow us to say something about ribbon knots!



What's "local"? How will we compute? The Będlewo Alexander formula: Let F be the faces of a knot diagram. Make an $F\times F$ matrix A by adding for each crossing contributions

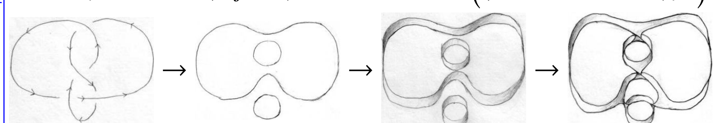


$$\to \begin{pmatrix} -1 & -1 & 2 & 0\\ 0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 1 & 0 & -1 & 0\end{pmatrix}$$

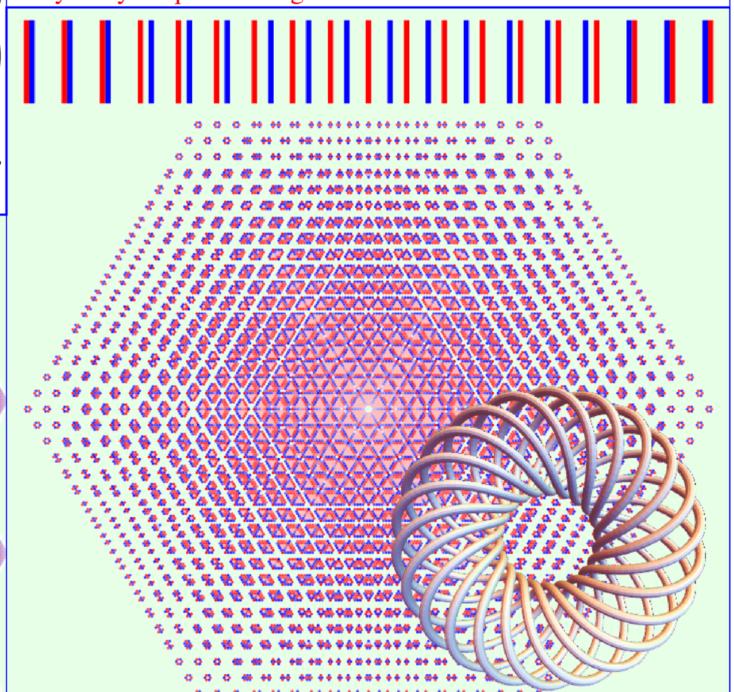


$$\to \begin{pmatrix} 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ -2 & 1 & 1 & 0\\ 1 & 0 & -1 & 0\end{pmatrix}$$

at rows / columns (i,j,k,l). Then $\Delta = \det'\left((T^{1/2}A - T^{-1/2}A)/2\right)$.


(the Seifert algorithm by Emily Redelmeier)

Expect the like for θ! Expect more like θ! Topology first! Resist the tyranny of quantum algebra!



2025/03/11@09:36

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Pitzer-250308.