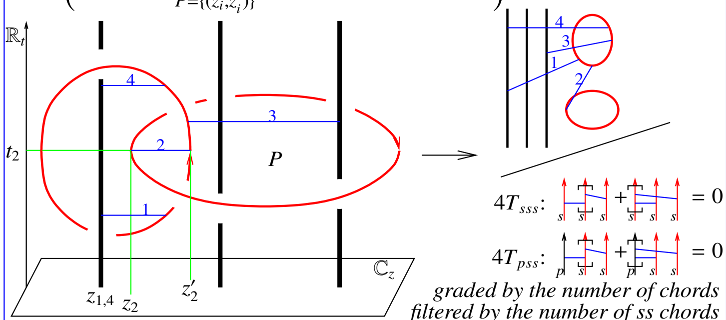
Kontsevich in a Pole Dance Studio. (w/o poles? See [Ko, BN])

$$Z=\left(\sum_{m=0}^{\infty}\frac{1}{(2\pi i)^m}\int_{\substack{t_1<\dots<t_m\\ P=\{(z_i,z_i')\}}}(-1)^{\#P_\downarrow}D_P\bigwedge_{i=1}^{m}\frac{dz_i-dz_i'}{z_i-z_i'}\right)^{\sim}\in\mathcal{A}$$



graded by the number of chords
filtered by the number of ss chords

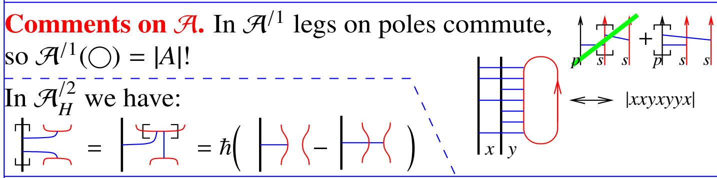
Comments on the Kontsevich Integral.

1. In the tangle case, the endpoints are fixed at top and bottom.
2. The $(\cdots)^{\sim}$ means "a correction is needed near the caps and the cups" (for the framed version, see [LM2, Da]).
3. There are never pp chords, and no $4T_{pps}$ and $4T_{ppp}$ relations.
4. Z is an "expansion".
5. Z respects the ss filtration and so descends to $Z^{/s}\colon\mathcal{K}^{/s}\to\mathcal{A}^{/s}$.

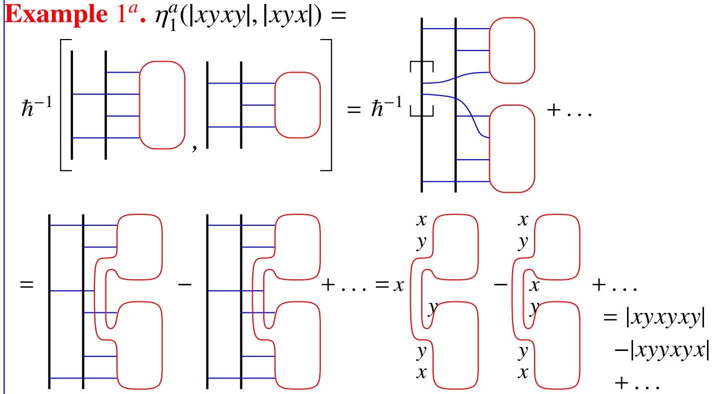
Comments on $\mathcal{A}$.

In $\mathcal{A}^{/1}$ legs on poles commute, so $\mathcal{A}^{/1}(\bigcirc)=|A|$!

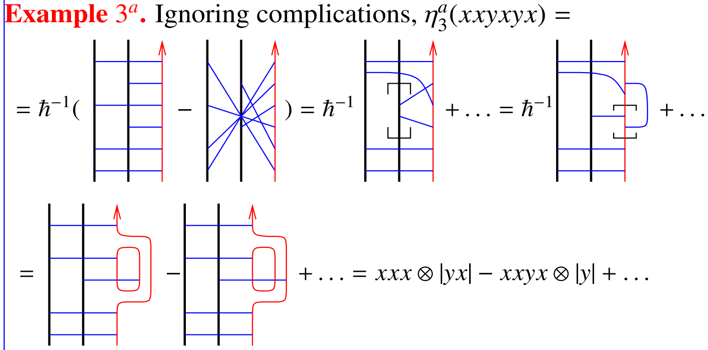
In $\mathcal{A}^{/2}_H$ we have:



Example 1^a. $\eta_1^a(|xyxy|,|xyx|)=$



Example 3^a. Ignoring complications, $\eta_3^a(xxyxyx)=$



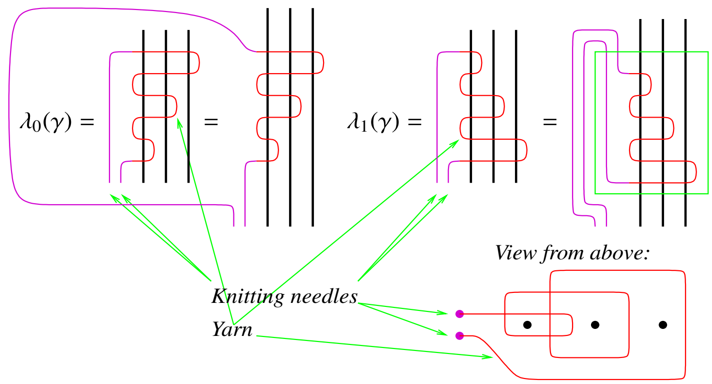
Proof of Lemma 1. We partially prove Theorem 2 instead:

Theorem 2. $\operatorname{gr}^\bullet\mathcal{K}_H\cong\mathbb{F}[\![\hbar]\!]\otimes(\mathcal{K}^{/1})_0$.

Proof mod $\hbar^2$. The map $\leftarrow$ is obvious. To go $\to$, map $\mathcal{K}_H\to\mathbb{F}[\![\hbar]\!]\otimes\mathcal{K}^{/1}$ using (positive crossing) $\mapsto$ (crossing) $+\frac{\hbar}{2}$)(and (negative crossing) $\mapsto$ (crossing) $-\frac{\hbar}{2}$)(and apply the functor $\operatorname{gr}^\bullet$.

Unignoring the Complications. We need λ_0 and λ_1 such that:

1. $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by flipping all self-intersections from ascending to descending.
2. Up to conjugation, $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by a global flip.
3. $Z(\lambda_i(\gamma))$ is computable from $W(\gamma)$ and $Z^{/1}(\lambda_i(\gamma))=W(\gamma)$.



Homework

1. Is there more than Examples 1–4?
2. Derive the bialgebra axioms from this perspective.
3. What more do we get if we don't mod out by HOMFLY-PT?
4. What more do we get if we allow more than one strand-strand interaction?
5. In this language, recover Kashiwara-Vergne [AKKN1, AKKN2].
6. How is all this related to w-knots?
7. Do the same with associators. Use that to derive formulas for solutions of Kashiwara-Vergne.
8. What's the relationship with the Habiro-Massuyeau invariants of links in handlebodies [HM] (different filtration!).
9. Pole dance on other surfaces!
10. Explore the action of the mapping class group.

Kashiwara Vergne

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Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-2208/