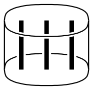
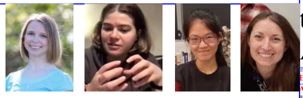
Tangles in a Pole Dance Studio: A Reading of Massuyeau, Alekseev, and Naef

Preliminary Definitions. Fix $p \in \mathbb{N}$ and $\mathbb{F} = \mathbb{Q}/\mathbb{C}$. Let $D_p := D^2\backslash(p \text{ pts})$, and let the Pole Dance Studio be $PDS_p := D_p \times I$.


PDS_3


Dancso Hogan Liu Scherich

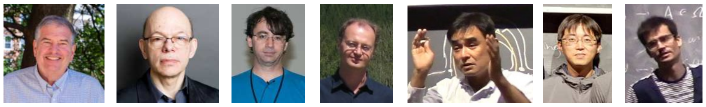
Abstract. I will report on joint work with Zsuzsanna Dancso, Tamara Hogan, Jessica Liu, and Nancy Scherich. Little of what we do is original, and much of it is simply a reading of Massuyeau [Ma] and Alekseev and Naef [AN1].

Jessica, Nancy, Tamara, Zsuzsi, & Dror in PDS_4

We study the pole-strand and strand-strand double filtration on the space of tangles in a pole dance studio (a punctured disk cross an interval), the corresponding homomorphic expansions, and a strand-only HOMFLY-PT relation. When the strands are transparent or nearly transparent to each other we recover and perhaps simplify substantial parts of the work of the aforementioned authors on expansions for the Goldman-Turaev Lie bi-algebra. ⟹

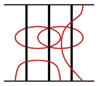
Expansions $W\colon FG\langle X_i\rangle \to FA\langle x_i\rangle$:
Magnus: $X_i \mapsto 1 + x_i$, $X_i^{-1} \mapsto 1 - x_i + x_i^2 - \dots$
Exponential: $X_i^{\pm 1} \mapsto e^{\pm x_i}$

Definitions. Let $\pi := FG\langle X_1, \dots, X_p\rangle$ be the free group (of deformation classes of based curves in D_p), $\bar\pi$ be the framed free group (deformation classes of based immersed curves), $|\pi|$ and $|\bar\pi|$ denote $\mathbb{F}$-linear combinations of cyclic words ($|x_iw| = |wx_i|$, unbased curves), $A := FA\langle x_1, \dots, x_p\rangle$ be the free associative algebra, and let $|A| := A/(x_iw = wx_i)$ denote cyclic algebra words.



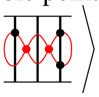
Theorem 1 (Goldman, Turaev, Massuyeau, Alekseev, Kawazumi, Kuno, Naef). $|\bar\pi|$ and $|A|$ are Lie bialgebras, and there is a "homomorphic expansion" $W\colon |\bar\pi| \to |A|$: a morphism of Lie bialgebras with $W(|X_i|) = 1 + |x_i| + \dots$.

Further Definitions. • $\mathcal{K} = \mathcal{K}_0 = \mathcal{K}_0^0 = \mathcal{K}(S) := \mathbb{F}\langle\text{framed tangles in } PDS_p\rangle$.



• $\mathcal{K}_t^s :=$(the image via ⤫ → ⤱ − ⤲ of tangles in PDS_p that have t double points, of which s are strand-strand).

E.g.,

$\mathcal{K}_5^2(\bigcirc) = \Big\langle$  $\Big\rangle$ /. ⤫ → ⤱ − ⤲

• $\mathcal{K}^{/s} := \mathcal{K}/\mathcal{K}^s$. Most important, $\mathcal{K}^{/1}(\bigcirc) = |\bar\pi|$, and there is $P\colon \mathcal{K}(\bigcirc) \to |\bar\pi|$.

• $\mathcal{A} := \prod \mathcal{K}_t/\mathcal{K}_{t+1}$, $\mathcal{A}^s := \prod \mathcal{K}_t^s/\mathcal{K}_{t+1}^s \subset \mathcal{A}$, $\mathcal{A}^{/s} := \mathcal{A}/\mathcal{A}^s$.

Fact 1. The Kontsevich Integral is an "expansion" $Z\colon \mathcal{K} \to \mathcal{A}$, compatible with several noteworthy structures.

Fact 2 (Le-Murakami, [LM1]). Z satisfies the strand-strand HOMFLY-PT relations: It descends to $Z_H\colon \mathcal{K}_H \to \mathcal{A}_H$, where

$$\mathcal{K}_H := \mathcal{K}\Big/\!\!\Big(⤱ - ⤲ = (e^{\hbar/2} - e^{-\hbar/2})\cdot \,)\,(\Big)$$

$$\mathcal{A}_H := \mathcal{A}/(\,\downarrow\!\!-\!\!\uparrow\; = \hbar\; ⤫ \;\text{ or }\; \uparrow\!\!-\!\!\uparrow\; = \hbar\; ⤫\,)$$

and $\deg\hbar = (1, 1)$.

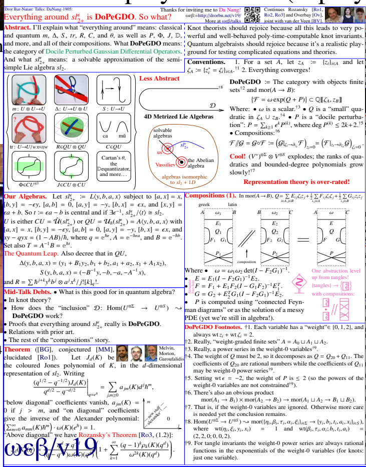
Proof of Fact 2. $Z(⤱) - Z(⤲) = ⤫ \cdot \left(e^{\mathrm{H}/2} - e^{-\mathrm{H}/2}\right)$

$$= ⤫ \cdot \left(e^{\hbar ⤫/2} - e^{-\hbar ⤫/2}\right) = \left(e^{\hbar/2} - e^{-\hbar/2}\right))\,(. \quad \square$$

Le, Murakami

Other Passions. With Roland van der Veen, I use "solvable approximation" and "Perturbed Gaussian Differential Operators" to unveil simple, strong, fast to compute, and topologically meaningful knot invariants near the Alexander polynomial. (⊂ polymath!)

van der Veen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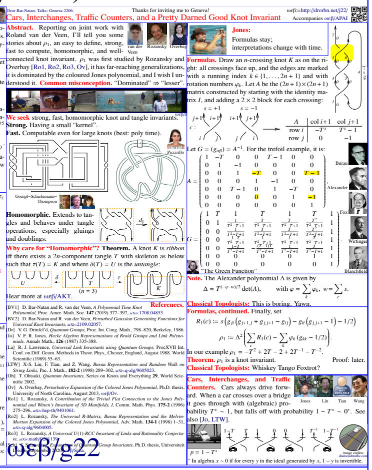
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Key 1. $W\colon |\bar\pi| \to |A|$ is $Z_H^{/1}\colon \mathcal{K}_H^{/1}(\bigcirc) \to \mathcal{A}_H^{/1}(\bigcirc)$.

Key 2 (Schematic). Suppose $\lambda_0, \lambda_1\colon |\bar\pi| \to \mathcal{K}(\bigcirc)$ are two ways of lifting plane curves into knots in PDS_p (namely, $P\circ\lambda_i = I$). Then for $\gamma \in |\bar\pi|$,

Lemma 1. "Division by $\hbar$" is well-defined.

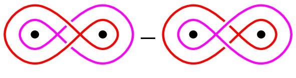
$$\eta(\gamma) := (\lambda_0(\gamma) - \lambda_1(\gamma))/\hbar \in \mathcal{K}_H^{/1}(\bigcirc\bigcirc) = |\bar\pi| \otimes |\bar\pi|$$

and we get an operation η on plane curves. If Kontsevich likes λ_0 and λ_1 (namely if there are λ_i^a with $Z^{/2}(\lambda_i(\gamma)) = \lambda_i^a(W(\gamma))$), then η will have a compatible algebraic companion η^a:

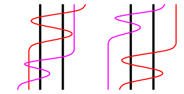
$$\eta^a(\alpha) := (\lambda_0^a(\alpha) - \lambda_1^a(\alpha))/\hbar \in \mathcal{A}_H^{/1}(\bigcirc\bigcirc) = |A| \otimes |A|.$$

For indeed, in $\mathcal{A}_H^{/2}$ we have $\hbar W(\eta(\gamma)) = \hbar Z(\eta(\gamma)) = Z(\lambda_0(\gamma)) - Z(\lambda_1(\gamma)) = \lambda_0^a(W(\gamma)) - \lambda_1^a(W(\gamma)) = \hbar\eta^a(W(\gamma))$.

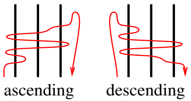
Example 1. With $\gamma_1, \gamma_2 \in |\pi|$ (or $|\bar\pi|$) set $\lambda_0(\gamma_1, \gamma_2) = \tilde\gamma_1 \cdot \tilde\gamma_2$ and $\lambda_1(\gamma_1, \gamma_2) = \tilde\gamma_2 \cdot \tilde\gamma_1$ where $\tilde\gamma_i$ are arbitrary lifts of γ_i. Then η_1 is the Goldman bracket! Note that here λ_0 and λ_1 are not well-defined, yet η_1 is.



Example 2. With $\gamma_1, \gamma_2 \in \pi$ (or $\bar\pi$) and with λ_0, λ_1 as on the right, we get the "double bracket" $\eta_2\colon \pi\otimes\pi \to \pi\otimes\pi$ (or $\bar\pi\otimes\bar\pi \to \bar\pi\otimes\bar\pi$).

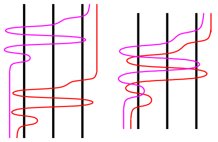


Example 3. With $\gamma \in \bar\pi$ and $\lambda_0(\gamma)$ its ascending realization as a bottom tangle and $\lambda_1(\gamma)$ its descending realization as a bottom tangle, we get $\eta_3\colon \bar\pi \to \bar\pi \otimes |\bar\pi|$. Closing the first component and anti-symmetrizing, this is the Turaev cobracket.


ascending descending

ascending descending

Example 4 [Ma]. With $\gamma \in \bar\pi$ and $\lambda_0(\gamma)$ its ascending outer double and $\lambda_1(\gamma)$ its ascending inner double we get $\eta_4\colon \bar\pi \to \bar\pi\otimes\bar\pi$. After some massaging, it too becomes the Turaev cobracket.



The rest is essentially **Exercises:** **1.** Lemma 1? **2.** $\mathcal{A}$? **3.** Fact 2? **4.** $\mathcal{A}^{/1}$? Especially, $\mathcal{A}^{/1}(\bigcirc) \cong |A|$! **5.** Explain why Kontsevich likes our λ's. **6.** Figure out η_i^a, $i = 1, \dots, 4$.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-2208/