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My talk yesterday:

More Dror: $\omega\epsilon\beta$ /talks

Dror Bar-Natan: Talks: Toronto-1912 [oeβ:=http://drorbn.net/to19/](http://drorbn.net/to19/)

Geography vs. Identity
Thanks for inviting me to the *Topology* session!

Abstract. Which is better, an emphasis on where things happen or on who are the participants? I can't tell; there are advantages and disadvantages either way. Yet much of quantum topology seems to be heavily and unfairly biased in favour of geography.

Geographers care for placement; for them, braids and tangles have ends at some distinguished points, hence they form categories whose objects are the placements of these points. For them, the basic operation is a binary "stacking of tangles". They are lead to monoidal categories, braided monoidal categories, representation theory, and much or most of we call "quantum topology".

Identifiers believe that strand identity persists even if one crosses or is being crossed. The key operation is a unary stitching operation m_c^{ab} , and one is lead to study meta-monoids, meta-Hopf-algebras, etc. See oeβ/leg, oeβ/kbh.

Braids.

Geography: $GB := \langle \gamma_i \rangle \left(\begin{matrix} \gamma_i \gamma_k = \gamma_k \gamma_i & \text{when } |i-k| > 1 \\ \gamma_i \gamma_{i+1} \gamma_i = \gamma_{i+1} \gamma_i \gamma_{i+1} \end{matrix} \right) = B$. (captures quantum algebra!)

Identity: $IB := \langle \sigma_{ij} \rangle \left(\begin{matrix} \sigma_{ij} \sigma_{kl} = \sigma_{kl} \sigma_{ij} & \text{when } \{|i, j, k, l\} = 4 \\ \sigma_{ij} \sigma_{jk} \sigma_{ik} = \sigma_{jk} \sigma_{ij} \sigma_{ik} & \text{when } \{|i, j, k\} = 3 \end{matrix} \right) = PB$.

Theorem. Let $S = \langle \tau \rangle$ be the symmetric group. Then \mathfrak{B} is both $PB \rtimes S \cong B * S \left(\begin{matrix} \gamma_i \tau = \tau \gamma_i & \text{when } \tau i = j, \tau(i+1) = (j+1) \end{matrix} \right)$ (and so PB is "bigger" than B , and hence quantum algebra doesn't see topology very well).

Proof. Going left, $\gamma_i \mapsto \sigma_{i,i+1}(i+1)$. Going right, if $i < j$ map $\sigma_{ij} \mapsto (j-1 \ j-2 \ \dots \ i) \gamma_{j-1}(i+1 \ \dots \ j)$ and if $i > j$ use $\sigma_{ij} \mapsto (j \ j+1 \ \dots \ i) \gamma_j(i-1 \ \dots \ j+1)$.

\mathfrak{B} views of σ_{ij} :

The Burau Representation of PB_n acts on $\mathbb{R}^n := \mathbb{Z}[t^{\pm 1}]^n = R(v_1, \dots, v_n)$ by $\sigma_{ij} v_k = v_k + \delta_{kj}(t-1)(v_j - v_i)$.

$\delta := \delta_{i,j} := \mathbf{1f}[i=j, 1, 0]$ oeβ/code
 $\mathfrak{B}_{i,j}[\mathcal{L}] := \mathcal{L} / v_n \mapsto v_n + \delta_{ij}(t-1)(v_j - v_i)$ // Expand
($\text{bas3} = \{v_2, v_3, v_4\}$) // $\mathfrak{B}_{1,2}$
 $\{v_2, v_1 - t v_2 + t v_3, v_3\}$
 $\text{bas3} // \mathfrak{B}_{1,2} // \mathfrak{B}_{1,3} // \mathfrak{B}_{2,3}$
 $\{v_2, v_1 - t v_2 + t v_3, v_1 - t v_4 + t v_2 - t^2 v_2 + t^2 v_3\}$
 $\text{bas3} // \mathfrak{B}_{2,3} // \mathfrak{B}_{1,3} // \mathfrak{B}_{1,2}$
 $\{v_2, v_1 - t v_2 + t v_3, v_1 - t v_4 + t v_2 - t^2 v_2 + t^2 v_3\}$

S_n acts on \mathbb{R}^n by permuting the v_i , so the Burau representation extends to \mathfrak{B}_n and restricts to B_n . With this, γ_i maps $v_i \mapsto v_{i+1}, v_{i+1} \mapsto \gamma_i(1-t)v_{i+1}$, and otherwise $v_k \mapsto v_k$.

Geography view: $\gamma_1 = \times \mid \mid \mid \quad \gamma_2 = \mid \times \mid \mid \quad \gamma_3 = \mid \mid \times \dots$
so x is γ_2 .

Identity view: At x strand 1 crosses strand 3, so x is σ_{13} .

The Gold Standard is set by the "T-calculus" Alexander formulas (oeβ/mac). An S -component tangle T has $\Gamma(T) \in R_S \times M_{S \times S}(R_S) = \begin{pmatrix} \omega & S \\ S & A \end{pmatrix}$ with $R_S := \mathbb{Z}\langle T_a : a \in S \rangle$:

$\begin{pmatrix} \omega & a & b & S \\ a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{pmatrix} \xrightarrow{m_c^{ab}} \begin{pmatrix} \omega & a & b & S \\ a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{pmatrix} \xrightarrow{T_a \sqcup T_b \rightarrow T_c} \begin{pmatrix} (1-\beta)\omega & c & S \\ c & \gamma + \frac{\alpha\beta}{1-\beta} & \epsilon + \frac{\alpha\beta}{1-\beta} \\ S & \phi + \frac{\alpha\beta}{1-\beta} & \Xi + \frac{\alpha\beta}{1-\beta} \end{pmatrix}$

The Gassner Representation of PB_n acts on $V = \mathbb{R}^n := \mathbb{Z}[t^{\pm 1}]^n = R(v_1, \dots, v_n)$ by $\sigma_{ij} v_k = v_k + \delta_{kj}(t-1)(v_j - v_i)$.

$\mathfrak{G}_{i,j}[\mathcal{L}] := \mathcal{L} / v_n \mapsto v_n + \delta_{ij}(t-1)(v_j - v_i)$ // Expand
($\text{bas3} // \mathfrak{G}_{1,2} // \mathfrak{G}_{1,3} // \mathfrak{G}_{2,3}$) // ($\text{bas3} // \mathfrak{G}_{1,2} // \mathfrak{G}_{1,3} // \mathfrak{G}_{2,3}$)

S_n acts on \mathbb{R}^n by permuting the v_i and the t_i , so the Gassner representation extends to \mathfrak{B}_n and then restricts to B_n as a \mathbb{Z} -linear ∞ -dimensional representation. It then descends to PB_n as a finite-rank R -linear representation, with lengthy non-local formulas.

Geographers: Gassner is an obscure partial extension of Burau.

Identifiers: Burau is a trivial silly reduction of Gassner.

The Turbo-Gassner Representation. With the same R and V , TG acts on $V \oplus (R^n \oplus \mathbb{Q}) \oplus (S^2 V \oplus V^n) = R(v_i, u_i, u_{ij}, w_{ij})$ by $\mathfrak{TG}_{i,j}[\mathcal{L}] := \mathcal{L} / \{$

$v_n \mapsto v_n + \delta_{ij}(t-1)(v_j - v_i) + v_{i,j} - v_{i,i} + \delta_{ij}(u_j - u_i) u_i w_{ij}$
 $v_{i,j} \mapsto v_{i,j} + (t-1) \times (\delta_{ij}(v_{i,j} - v_{i,i}) + (\delta_{i,i} - \delta_{i,j}) t_i^2 t_j)$
 $(u_i + \delta_{ij}(t-1)(u_j - u_i)) u_i w_{ij}$
 $u_i \mapsto u_i + \delta_{ij}(t-1)(u_j - u_i)$
 $w_{ij} \mapsto w_{ij} + (\delta_{ij} - \delta_{ij}) (t_i^2 - 1) w_{ij}$ // Expand
 $\text{bas3} = \{v_2, v_3, u_2^2 w_2, u_2^2 w_3, v_{2,2}, v_{2,3}, v_{3,2}, v_{3,3}, u_2^2 w_2, u_2^2 w_3, u_1 u_2 w_2, u_1 u_2 w_3, u_2 u_3 w_2, u_2 u_3 w_3, u_1^2 w_2, u_1^2 w_3\}$
($\text{bas3} // \mathfrak{TG}_{1,2} // \mathfrak{TG}_{1,3} // \mathfrak{TG}_{2,3}$) // ($\text{bas3} // \mathfrak{TG}_{1,2} // \mathfrak{TG}_{1,3} // \mathfrak{TG}_{2,3}$)

True
Like Gassner, TG is also a representation of PB_n .

I have no idea where it belongs!

My talk tomorrow, at the *chord diagrams everywhere* session: oeβ/talks

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