

```

Define [op_is__ = E_] :=
Module [ {SD, ii, jj, kk, isp, nis, nisp, sis},
Block [ {i, j, k},
ReleaseHold [Hold [
SD [op_nisp, $k_Integer, Block [ {i, j, k}, op_isp, $k = E;
op_nis, $k ]];
SD [op_isp, op [is], $k]; SD [op_sis__, op [sis]];
] /. {SD -> SetDelayed,
isp -> {is} /. {i -> ii, j -> jj, k -> kk},
nis -> {is} /. {i -> ii, j -> jj, k -> kk},
nisp -> {is} /. {i -> ii_, j -> jj_, k -> kk_}
} ] ] ]

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### The Fundamental Tensors

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Define [am_i,j-k = E [i,j]->{k} [ (alpha_i + alpha_j) a_k, (e^{-gamma alpha_j} xi_i + xi_j) x_k, 1 ] $k,
bm_i,j-k = E [i,j]->{k} [ (beta_i + beta_j) b_k, (eta_i + eta_j) y_k, e^{(e^{-beta_i} - 1) eta_j y_k} ] $k ]
Define [R_i,j =
E [i]->{i,j} [ h a_j b_i, h x_j y_i, e^{(sum_{k=2}^{j-k+1} (1 - e^{gamma e^h})^k (h y_i x_j)^k) / k (1 - e^{gamma e^h})) } ] $k ]
Define [R_bar_i,j = E [i]->{i,j} [ -h a_j b_i, -h x_j y_i / B_i,
1 + If [ $k == 0, 0, (R_bar [i,j], $k-1) $k [3] -
((R_bar [i,j], 0) $k R_{1,2} (R_bar [3,4], $k-1) $k) // (bm_{i,1-i} am_{j,2-j}) //
(bm_{i,3-i} am_{j,4+j}) [3] ] ],
P_i,j = E [i,j]->{} [ beta_i alpha_j / h, eta_i xi_j / h,
1 + If [ $k == 0, 0, (P [i,j], $k-1) $k [3] -
(R_{1,2} // ((P [3,j], 0) $k (P [i,2], $k-1) $k)) [3] ] ] ]
Define [a_Sj = R_bar_i,j ~ B_i ~ P_i,j,
a_bar_Si = E [i]->{i} [ -a_i alpha_i, -x_i xi_i xi_i,
1 + If [ $k == 0, 0, (a_bar_S [i], $k-1) $k [3] -
((a_bar_S [i], 0) $k ~ B_i ~ a_Si ~ B_i ~ (a_bar_S [i], $k-1) $k) [3] ] ] ]
Define [b_Si = R_i,1 ~ B_1 ~ a_S1 ~ B_1 ~ P_i,1,
b_bar_Si = R_i,1 ~ B_1 ~ a_bar_S1 ~ B_1 ~ P_i,1,
a_Delta_i-j,k = (R_{1,j} R_{2,k}) // bm_{1,2-3} // P_{3,i},
b_Delta_i-j,k = (R_{j,1} R_{k,2}) // am_{1,2-3} // P_{i,3} ]

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Define [
dm_i,j-k =
(E [i,j]->{i,j} [ beta_i b_i + alpha_j a_j, eta_i y_i + xi_j x_j, 1 ]
(a_Delta_{i-1,2} // a_Delta_{2-3} // a_bar_S3) (b_Delta_{j-1,-2} // b_Delta_{-2,-3}) //
(P_{-1,3} P_{-3,1} am_{2,j-k} bm_{i,-2-k}),
dSi = E [i]->{1,2} [ beta_i b_1 + alpha_i a_2, eta_i y_1 + xi_i x_2, 1 ] // (b_bar_S1 a_S2) //
dm_{2,1-i},
dDelta_i-j,k = (b_Delta_{i-3,1} a_Delta_{i-2,4}) // (dm_{3,4-k} dm_{1,2-j}) ]
Define [Ci = E [i]->{i} [ 0, 0, B_i^{1/2} e^{-h e^{alpha_i} / 2} ] $k,
C_bar_i = E [i]->{i} [ 0, 0, B_i^{-1/2} e^{h e^{alpha_i} / 2} ] $k,
Kink_i = (R_{1,3} C2) // dm_{1,2+i} // dm_{1,3-i},
Kink_bar_i = (R_bar_{1,3} C2) // dm_{1,2+i} // dm_{1,3-i} ]
Define [
b2t_i = E [i]->{i} [ alpha_i a_i - beta_i t_i / gamma, xi_i x_i + eta_i y_i, e^{e^{beta_i} a_i / gamma} ] $k,
t2b_i = E [i]->{i} [ alpha_i a_i - tau_i gamma b_i, xi_i x_i + eta_i y_i, e^{e^{tau_i} a_i} ] $k ]
Define [kR_i,j = R_i,j // (b2t_i b2t_j) /. {t_i | j -> t,
kR_bar_i,j = R_bar_i,j // (b2t_i b2t_j) /. {t_i | j -> t, T_i | j -> T},
km_i,j-k = (t2b_i t2b_j) // dm_{i,j-k} //
b2t_k /. {t_k -> t, T_k -> T, tau_i | j -> 0},
kCi = Ci // b2t_i /. T_i -> T, kC_bar_i = C_bar_i // b2t_i /. T_i -> T,
kKink_i = Kink_i // b2t_i /. {t_i -> t, T_i -> T},
kKink_bar_i = Kink_bar_i // b2t_i /. {t_i -> t, T_i -> T} ]

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### The Trefoil

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$K = 2; Z = kR_{1,5} kR_{6,2} kR_{3,7} kC4 kKink8 kKink9 kKink10;
Do [ Z = Z ~ B_{1,r} ~ km_{1,r-1}, {r, 2, 10} ];
Simplify [ @Z /. v_{-1} -> v
E [i]->{1} [ 0, 0, 1 / (1 - T + T^2) + 1 / (1 - T + T^2)^3 T h (2 a (-1 + T - T^3 + T^4) +
T (-1 + 2 T - 3 T^2 + 2 T^3) gamma - 2 (1 + T^3) x y gamma h) e +
1 / (2 (1 - T + T^2)^5) T h^2 (4 a^2 (1 - T + T^2)^2 (1 + T - 6 T^2 + T^3 + T^4) +
4 a (1 - T + T^2) gamma (T (2 - 5 T + 8 T^2 - 7 T^3 - 2 T^4 + 2 T^5) -
2 (-1 - 2 T + 5 T^2 - 4 T^3 + T^4 + 2 T^5) x y h) +
gamma^2 (T (1 - 2 T + 4 T^2 - 2 T^3 + 6 T^5 - 11 T^6 + 4 T^7) +
4 (-1 + 2 T + T^3 + T^4 + 2 T^6 - T^7) x y h +
6 (1 - T + T^2)^2 (1 + 3 T + T^2) x^2 y^2 h^2) ] e^2 + 0 [e]^3 ]

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diagram	$n_k^i$ Today's $\rho_k^+$	Alexander's $\omega^+$ unknotting # / amphi?	genus / ribbon	diagram	$n_k^i$ Today's $\rho_k^+$	Alexander's $\omega^+$ unknotting # / amphi?	genus / ribbon	diagram	$n_k^i$ Today's $\rho_k^+$	Alexander's $\omega^+$ unknotting # / amphi?	genus / ribbon
	$0_1^1$ 0	1	0 / ✓ 0 / ✓		$3_1^1$ $t$	$t - 1$	1 / ✗ 1 / ✗		$4_1^1$ 0	$3 - t$	1 / ✗ 1 / ✓
	$5_1^1$ $2t^3 + 3t$	$t^2 - t + 1$	2 / ✗ 2 / ✗		$5_2^1$ $5t - 4$	$2t - 3$	1 / ✗ 1 / ✗		$6_1^1$ $t - 4$	$5 - 2t$	1 / ✓ 1 / ✗
	$6_2^1$ $t^3 - 4t^2 + 4t - 4$	$-t^2 + 3t - 3$	2 / ✗ 1 / ✗		$6_3^1$ 0	$t^2 - 3t + 5$	2 / ✗ 1 / ✓		$7_1^1$ $3t^5 + 5t^3 + 6t$	$t^3 - t^2 + t - 1$	3 / ✗ 3 / ✗
	$7_2^1$ $14t - 16$	$3t - 5$	1 / ✗ 1 / ✗		$7_3^1$ $-9t^3 + 8t^2 - 16t + 12$	$2t^2 - 3t + 3$	2 / ✗ 2 / ✗		$7_4^1$ $32 - 24t$	$4t - 7$	1 / ✗ 2 / ✗
	$7_5^1$ $9t^3 - 16t^2 + 29t - 28$	$2t^2 - 4t + 5$	2 / ✗ 2 / ✗		$7_6^1$ $t^3 - 8t^2 + 19t - 20$	$-t^2 + 5t - 7$	2 / ✗ 1 / ✗		$7_7^1$ $8 - 3t$	$t^2 - 5t + 9$	2 / ✗ 1 / ✗
	$8_1^1$ $5t - 16$	$7 - 3t$	1 / ✗ 1 / ✗		$8_2^1$ $-2t^5 + 8t^4 + 10t^3 - 12t^2 + 13t - 12$	$-t^3 + 3t^2 - 3t + 3$	3 / ✗ 2 / ✗		$8_3^1$ 0	$9 - 4t$	1 / ✗ 2 / ✓
	$8_4^1$ $3t^3 - 8t^2 + 6t - 4$	$-2t^2 + 5t - 5$	2 / ✗ 2 / ✗		$8_5^1$ $-2t^5 + 8t^4 - 13t^3 + 20t^2 - 22t + 24$	$-t^3 + 3t^2 - 4t + 5$	3 / ✗ 2 / ✗		$8_6^1$ $5t^3 - 20t^2 + 28t - 32$	$-2t^2 + 6t - 7$	2 / ✗ 2 / ✗
	$8_7^1$ $-t^5 + 4t^4 - 10t^3 + 12t^2 - 13t + 12$	$t^3 - 3t^2 + 5t - 5$	3 / ✗ 1 / ✗		$8_8^1$ $-t^3 + 4t^2 - 12t + 16$	$2t^2 - 6t + 9$	2 / ✓ 2 / ✗		$8_9^1$ 0	$-t^3 + 3t^2 - 5t + 7$	3 / ✓ 1 / ✓
	$8_{10}^1$ $-t^5 + 4t^4 - 11t^3 + 16t^2 - 21t + 20$	$t^3 - 3t^2 + 6t - 7$	3 / ✗ 2 / ✗		$8_{11}^1$ $5t^3 - 24t^2 + 39t - 44$	$-2t^2 + 7t - 9$	2 / ✗ 1 / ✗		$8_{12}^1$ 0	$t^2 - 7t + 13$	2 / ✗ 2 / ✓
	$8_{13}^1$ $-t^3 + 4t^2 - 14t + 20$	$2t^2 - 7t + 11$	2 / ✗ 1 / ✗		$8_{14}^1$ $5t^3 - 28t^2 + 57t - 68$	$-2t^2 + 8t - 11$	2 / ✗ 1 / ✗		$8_{15}^1$ $21t^3 - 64t^2 + 120t - 140$	$3t^2 - 8t + 11$	2 / ✗ 2 / ✗
	$8_{16}^1$ $t^5 - 6t^4 + 17t^3 - 28t^2 + 35t - 36$	$t^3 - 4t^2 + 8t - 9$	3 / ✗ 2 / ✗		$8_{17}^1$ 0	$-t^3 + 4t^2 - 8t + 11$	3 / ✗ 1 / ✓		$8_{18}^1$ 0	$-t^3 + 5t^2 - 10t + 13$	3 / ✗ 2 / ✓
	$8_{19}^1$ $-3t^5 - 4t^2 - 3t$	$t^3 - t^2 + 1$	3 / ✗ 3 / ✗		$8_{20}^1$ $4t - 4$	$t^2 - 2t + 3$	2 / ✓ 1 / ✗		$8_{21}^1$ $t^3 - 8t^2 + 16t - 20$	$-t^2 + 4t - 5$	2 / ✓ 1 / ✗