

The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(\mathfrak{g})$ or $\hat{\mathcal{U}}_q(\mathfrak{g})$) and elements

$$R = \sum a_i \otimes b_i \in U \otimes U \quad \text{and} \quad C \in U,$$

form

$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

Problem. Extract information from Z .
The Dogma. Use representation theory. In principle finite, but *slow*.

The (fake) moduli of Lie algebras on V , a quadratic variety in $(V^*)^{\otimes 2} \otimes V$ is on the right. We care about $sl_{17}^\epsilon := sl_{17}^\epsilon / (\epsilon^{k+1} = 0)$.

Solvable Approximation. A quantized universal enveloping algebra (aka “quantum group”) is an ∞ -dimensional inverse limit.

untrimmed $\longrightarrow$ halfway-trimmed $\longrightarrow$ almost fully-trimmed

too hard $\longleftarrow$ “solvable approximation” (a parameter is hidden) $\longleftarrow$ “finite-type”

Recomposing gl_n . Half is enough! $gl_n \oplus a_n = \mathcal{D}(\nabla, b, \delta)$:

$b(\nabla) = b: \nabla \otimes \nabla \rightarrow \nabla$
 $b(\Delta) \sim \delta: \nabla \rightarrow \nabla \otimes \nabla$

Now define $gl_n^\epsilon := \mathcal{D}(\nabla, b, \epsilon\delta)$. Schematically, this is $[\nabla, \nabla] = \nabla$, $[\Delta, \Delta] = \epsilon\Delta$, and $[\nabla, \Delta] = \Delta + \epsilon\nabla$. In detail, it is

$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj} \quad [f_{ij}, f_{kl}] = \epsilon \delta_{jk} f_{il} - \epsilon \delta_{li} f_{kj}$
 $[e_{ij}, f_{kl}] = \delta_{jk} (\epsilon \delta_{i < k} e_{il} + \delta_{il} (h_i + \epsilon g_i) / 2 + \delta_{i > l} f_{il})$
 $\quad - \delta_{li} (\epsilon \delta_{k < j} e_{kj} + \delta_{kj} (h_j + \epsilon g_j) / 2 + \delta_{k > j} f_{kj})$
 $[g_i, e_{jk}] = (\delta_{ij} - \delta_{ik}) e_{jk} \quad [h_i, e_{jk}] = \epsilon (\delta_{ij} - \delta_{ik}) e_{jk}$
 $[g_i, f_{jk}] = (\delta_{ij} - \delta_{ik}) f_{jk} \quad [h_i, f_{jk}] = \epsilon (\delta_{ij} - \delta_{ik}) f_{jk}$

Solvable Approximation (2). At $\epsilon = 1$ and modulo $h = g$, the above is just gl_n . By rescaling at $\epsilon \neq 0$, gl_n^ϵ is independent of ϵ . We let gl_n^ϵ be gl_n^ϵ regarded as an algebra over $\mathbb{Q}[\epsilon]/\epsilon^{k+1} = 0$. It is the “ k -smidgen solvable approximation” of gl_n !

Recall that $\mathfrak{g}$ is “solvable” if iterated commutators in it ultimately vanish: $\mathfrak{g}_2 := [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_3 := [\mathfrak{g}_2, \mathfrak{g}_2]$, $\dots$, $\mathfrak{g}_d = 0$. Equivalently, if it is a subalgebra of some large-size ∇ algebra.

Note. This whole process makes sense for arbitrary semi-simple Lie algebras.

GDO-Categories. Given $\mathfrak{g}$ with basis $B = \{x, y, \dots\}$, consider the following diagram:

$$\begin{array}{ccccc} \mathbb{Q} = \hat{\mathcal{U}}_{(q)}(\bigoplus_0 \mathfrak{g}) & \xrightarrow{Z} & \hat{\mathcal{U}}_{(q)}(\mathfrak{g}) & \xrightleftharpoons[m]{\Delta} & \hat{\mathcal{U}}_{(q)}(\bigoplus_2 \mathfrak{g}) \\ & \uparrow \scriptstyle \mathbb{O}(xy, \dots; \cdot) & \uparrow \scriptstyle \mathbb{O}(yx, \dots; \cdot) & & \uparrow \scriptstyle \mathbb{O}(y_1 x_1 \dots y_2 x_2 \dots; \cdot) \\ \hat{S}(\emptyset) & \xrightarrow{Z} & \hat{S}(B) & \xrightleftharpoons[m]{\Delta} & \hat{S}(B_1, B_2) \\ & & \downarrow \scriptstyle SW_{xy} & & \end{array}$$

Hence Z , SW_{xy} , m , Δ , (and likewise S and θ) are morphisms in the completion of the monoidal category $\mathcal{F}$ whose objects are finite sets B and whose morphisms are $\text{mor}_{\mathcal{F}}(B, B') := \text{Hom}_{\mathbb{Q}}(S(B) \rightarrow S(B')) = S(B^*, B')$ (by convention, $x^* = \xi$, $y^* = \eta$, etc.). Ergo we need to *consolidate* (at least parts of) said completion.

Definition. A “docile perturbed Gaussian” in the variables $(z_i)_{i \in S}$ over the ring R is an expression of the form

$$\mathbb{E}^{q^{ij} z_i z_j} P = \mathbb{E}^{q^{ij} z_i z_j} \left(\sum_{k \geq 0} \epsilon^k P_k \right),$$

where all coefficients are in R and where P is a “docile series”: $\deg P_k \leq 4k$.

Docility Matters! The rank of the space of docile series to ϵ^k is polynomial in the number of variables $|S|$.

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K , in the d -dimensional representation of sl_2 . Writing

$$\frac{(q^{1/2} - q^{-1/2}) J_d(K)}{q^{d/2} - q^{-d/2}} \Big|_{q=e^h} = \sum_{j,m \geq 0} a_{jm}(K) d^j h^m,$$

“below diagonal” coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and “on diagonal” coefficients give the inverse of the Alexander polynomial: $(\sum_{m=0}^{\infty} a_{mm}(K) h^m) \cdot \omega(K)(e^h) = 1$.

“Above diagonal” we have **Rozansky’s Theorem** [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1}) \omega(K)(q^d)} \left(1 + \sum_{k=1}^{\infty} \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)} \right).$$

Prior art. Some amazing computations by Rozansky and Overbay in [Ro2, Ro3] and in [Ov].

Faddeev’s Formula (In as much as we can tell, first appeared w/o proof in Faddeev [Fa], rediscovered and proven in Quesne [Qu], and again with easier proof, in Zagier [Za]). With $[n]_q := \frac{q^n - 1}{q - 1}$, with $[n]_q! := [1]_q [2]_q \cdots [n]_q$ and with $\mathbb{E}_q^x := \sum_{n \geq 0} \frac{x^n}{[n]_q!}$, we have

$$\log \mathbb{E}_q^x = \sum_{k \geq 1} \frac{(1-q)^k x^k}{k(1-q^k)} = x + \frac{(1-q)^2 x^2}{2(1-q^2)} + \dots$$

Proof. We have that $\mathbb{E}_q^x = \frac{\mathbb{E}_q^{qx} - \mathbb{E}_q^x}{qx - x}$ (“the q -derivative of $\mathbb{E}_q^x$ is itself”), and hence $\mathbb{E}_q^{qx} = (1 + (1-q)x) \mathbb{E}_q^x$, and

$$\log \mathbb{E}_q^{qx} = \log(1 + (1-q)x) + \log \mathbb{E}_q^x.$$

Writing $\log \mathbb{E}_q^x = \sum_{k \geq 1} a_k x^k$ and comparing powers of x , we get $q^k a_k = -(1-q)^k / k + a_k$, or $a_k = \frac{(1-q)^k}{k(1-q^k)}$. $\square$

Aside. “Consolidate” means “give a finite name to an infinite object, and figure out how to sufficiently manipulate such finite names”. E.g., solving $f'' = -f$ we encounter and set $\sum \frac{(-1)^k x^{2k}}{(2k)!} \leadsto \cos x$, $\sum \frac{(-1)^k x^{2k+1}}{(2k+1)!} \leadsto \sin x$, and then $\cos^2 x + \sin^2 x = 1$ and $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

The Composition Law. If

$$S(B_0) \xrightarrow[f \in \mathbb{Q}[\zeta_{0i}, \zeta_{1j}]]{f} S(B_1) \xrightarrow[g \in \mathbb{Q}[\zeta_{1j}, \zeta_{2k}]]{g} S(B_2)$$

$$\text{then } {}^t(f \parallel g) = {}^t(g \circ f) = \left(g|_{\zeta_{1j} \rightarrow \partial_{\zeta_{1j}}} f \right)_{\zeta_{1j}=0}.$$

Examples.

1. The 1-variable identity map $I: S(z) \rightarrow S(z)$ is given by ${}^t I_1 = \mathbb{E}^{z\zeta}$ and the n -variable one by ${}^t I_n = \mathbb{E}^{z_1 \zeta_1 + \dots + z_n \zeta_n}$.