

1-Smidgen sl_2 Let \mathfrak{g}_1 be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b, c, u, w \rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and with $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$, with CYBE $r_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes(i,j)}$. Over \mathbb{Q} , \mathfrak{g}_1 is a **solvable approximation of sl_2** : $\mathfrak{g}_1 \supset \langle b, u, w, \epsilon b, \epsilon c, \epsilon u, \epsilon w \rangle \supset \langle b, \epsilon b, \epsilon c, \epsilon u, \epsilon w \rangle \supset 0$. (note: $\deg(b, c, u, w, \epsilon) = (1, 0, 1, 0, 1)$)

0-Smidgen $sl_2 \odot$. Let \mathfrak{g}_0 be \mathfrak{g}_1 at $\epsilon = 0$, or $\mathbb{Q}\langle b, c, u, w \rangle / ([b, \cdot] = 0, [c, u] = u, [c, w] = -w, [u, w] = b$ with $r_{ij} = b_i c_j + u_i w_j$. It is $\mathfrak{b}^* \rtimes \mathfrak{b}$ where \mathfrak{b} is the 2D Lie algebra $\mathbb{Q}\langle c, w \rangle$ and (b, u) is the dual basis of (c, w) . For topology, it is more valuable than \mathfrak{g}_1 / sl_2 , but topology already got by other means almost everything \mathfrak{g}_0 gives.

How did these arise? $sl_2 = \mathfrak{b}^+ \oplus \mathfrak{b}^- / \mathfrak{h} =: sl_2^+ / \mathfrak{h}$, where $\mathfrak{b}^+ = \langle c, w \rangle / [w, c] = w$ is a Lie bialgebra with $\delta: \mathfrak{b}^+ \rightarrow \mathfrak{b}^+ \otimes \mathfrak{b}^+$ by $\delta: (c, w) \mapsto (0, c \wedge w)$. Going back, $sl_2^+ = \mathcal{D}(\mathfrak{b}^+) = (\mathfrak{b}^+)^* \oplus \mathfrak{b}^+ = \langle b, u, c, w \rangle / \dots$. **Idea.** Replace $\delta \rightarrow \epsilon \delta$ over $\mathbb{Q}[\epsilon]/(\epsilon^{k+1} = 0)$. At $k = 0$, get \mathfrak{g}_0 . At $k = 1$, get $[w, c] = w$, $[w, b'] = -\epsilon w$, $[c, u] = u$, $[b', u] = -\epsilon u$, $[b', c] = 0$, and $[u, w] = b' - \epsilon c$. Now note that $b' + \epsilon c$ is central, so switch to $b := b' + \epsilon c$. This is \mathfrak{g}_1 .

Ordering Symbols. \odot (*poly* | *specs*) plants the variables of *poly* in $\mathcal{S}(\otimes \mathfrak{g}_i)$ on several tensor copies of $\mathcal{U}(\mathfrak{g})$ according to *specs*. E.g., $\odot(c_1^3 u_1 c_2 e^{u_3} w_3^9 | x: w_3 c_1, y: u_1 u_3 c_2) = w^9 c^3 \otimes u e^u c \in \mathcal{U}(\mathfrak{g})_x \otimes \mathcal{U}(\mathfrak{g})_y$. This enables the description of elements of $\hat{\mathcal{U}}(\mathfrak{g})^{\otimes S}$ using commutative polynomials / power series.

0-Smidgen Invariants. $r = Id \in \mathfrak{b}^- \otimes \mathfrak{b}^+$ solves the CYBE $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$ in $\mathcal{U}(\mathfrak{g}_0)^{\otimes 3}$ and, by luck,

$$\begin{array}{c} \nearrow \\ + \\ i \end{array} \begin{array}{c} \searrow \\ - \\ j \end{array} = \begin{array}{c} \uparrow \\ + \\ i \end{array} \begin{array}{c} \uparrow \\ + \\ j \end{array} = R_{ij} = e^{r_{ij}} = e^{b_i c_j + u_i w_j} \in \mathcal{U}(\mathfrak{g}_{0,i} \oplus \mathfrak{g}_{0,j})$$

solves YB/R3.

Lemma. $R_{ij} = e^{b_i c_j + u_i w_j} = \odot(\exp(b_i c_j + \frac{e^{b_i} - 1}{b_i} u_i w_j) | i: u_i, j: c_j w_j)$

Example. $Z(T_0) = \sum_{m,n} \frac{b_i^{m-n} (e^{b_i} - 1)^n}{m! n!} u^n \otimes c^m w^n$.

$$\odot(\exp(b_5 c_1 + \frac{e^{b_5} - 1}{b_5} u_5 w_1 + b_2 c_4 + \frac{e^{b_2} - 1}{b_2} u_2 w_4 - b_3 c_6 + \frac{e^{-b_3} - 1}{b_3} u_3 w_6) | \text{“ucw form”} \\ x: c_1 w_1 u_2, y: u_3 c_4 w_4 u_5 c_6 w_6) = \odot(\zeta | x: u_x c_x w_x, y: u_y c_y w_y)$$

Goal. Write ζ as a Gaussian: ωe^{L+Q} where L bilinear in b_i and c_i with integer coefficients, Q a balanced quadratic in u_i and w_i with coefficients in $R_S := \mathbb{Q}(b_i, e^{b_i})$, and $\omega \in R_S$.

The Big \mathfrak{g}_0 Lemma. Under $[c, u] = u$, $[c, w] = -w$, and $[u, w] = b$:

- 1a. $N^{cu} := \odot(e^{\gamma c + \beta u} | uc) \xrightarrow{\gamma} \odot(e^{\gamma c + e^{\beta} \beta u} | cu)$ (means $e^{\beta u} e^{\gamma c} = e^{\gamma c} e^{\beta u}$)
- 1b. $N^{wc} := \odot(e^{\gamma c + \alpha w} | wc) \xrightarrow{\gamma} \odot(e^{\gamma c + e^{\alpha} \alpha w} | cw)$... in the $(ax + b)$ group)
2. $\odot(e^{\alpha w + \beta u} | wu) = \odot(e^{-b\alpha\beta + \alpha w + \beta u} | uw)$ (the Weyl relations)
3. $\odot(e^{\delta u w} | wu) e^{\beta u} = e^{\gamma \beta u} \odot(e^{\delta u w} | wu)$, with $\gamma = (1 + b\delta)^{-1}$
- (a. expand and crunch. b. use $w = b\hat{x}$, $u = \partial_x$. c. use “scatter and glow”.)
4. $\odot(e^{\delta u w} | wu) = \odot(\gamma e^{\gamma \delta u w} | uw)$ (same techniques)
5. $N^{wu} := \odot(e^{\beta u + \alpha w + \delta u w} | wu) \xrightarrow{\beta} \odot(\gamma e^{-b\alpha\beta + \gamma\alpha w + \gamma\beta u + \gamma\delta u w} | uw)$
6. $N_k^{c_i c_j} := \odot(\zeta | c_i c_j) \xrightarrow{\gamma} \odot(\zeta / (c_i, c_j \rightarrow c_k) | c_k)$

Sneaky. α may contain (other) u 's, β may contain (other) w 's.

Strand Stitching. m_k^{ij} is defined as the composition

$$u_i c_i \overline{w_i u_j} c_j w_j \xrightarrow{N_x^{w_i u_j}} u_i \overline{c_i u_x} \overline{w_x c_j} w_j \xrightarrow{N_x^{c_i u_x} // N_x^{w_x c_j}} \overline{u_i u_x} \overline{c_x c_x} \overline{w_x w_x} w_j \xrightarrow{i,j,x \rightarrow k} u_k c_k w_k$$

On to 1-smidgen invariants, where much is the same. . .

The Big \mathfrak{g}_1 Lemma. Parts 1 and 6 are the same, yet
 5. $\odot(e^{\alpha w + \beta u + \delta u w} | wu) = \odot(\gamma(1 + \epsilon v \Lambda) e^{\gamma(-b\alpha\beta + \alpha w + \beta u + \delta u w)} | ucw)$
 Here Λ is for $\Lambda\acute{o}\gamma\omicron\varsigma$, “a principle of order and knowledge”, a balanced quartic in α, β, u, c , and w :

$$\begin{aligned} \Lambda = & -b\gamma(\alpha^2 \beta^2 \gamma^2 + 4\alpha\beta\delta\gamma + 2\delta^2)/2 + \beta^2 \delta \gamma^3 (b\delta + 2)u^2/2 \\ & + \delta^3 \gamma^3 (3b\delta + 4)u^2 w^2/2 + \beta \delta^2 \gamma^3 (2b\delta + 3)u^2 w \\ & + \alpha \delta^2 \gamma^3 (2b\delta + 3)u w^2 + 2\delta \gamma^2 (b\delta + 2)(\alpha\beta\gamma + \delta)u w \\ & + \alpha^2 \delta \gamma^3 (b\delta + 2)w^2/2 + 2(\alpha\beta\gamma + \delta)c + 2\beta\delta\gamma u c + 2\delta^2 \gamma u c w \\ & + 2\alpha\delta\gamma c w + \beta\gamma^2 (\alpha\beta\gamma + 2\delta)u + \alpha\gamma^2 (\alpha\beta\gamma + 2\delta)w. \end{aligned}$$

Proof. A lengthy computation. (Verification: $\omega\epsilon\beta/\text{Big}$)

Problem. We now need to normal-order perturbed Gaussians!

Solution. Borrow some tactics from QFT:

$$\odot(\epsilon P(c, u) e^{\gamma c + \beta u} | uc) = \odot(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + \beta u} | uc) = \odot(\epsilon P(\partial_\gamma, \partial_\beta) e^{\gamma c + e^{-\gamma} \beta u} | cu),$$

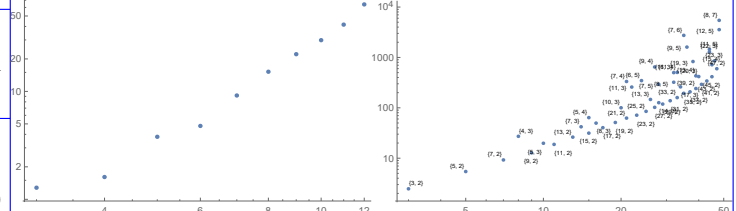
$$\odot(\epsilon P(u, w) e^{\alpha w + \beta u + \delta u w} | wu) = \odot(\epsilon P(\partial_\beta, \partial_\alpha) \gamma e^{\gamma(-b\alpha\beta + \alpha w + \beta u + \delta u w)} | ucw)$$

Finally, the values of the generators $\nearrow, \searrow, \vec{n}$, and \underline{u} , are set by solving many equations, non-uniquely.

Pragmatic Simplifications. Set $t := e^b$, work with $v := (t-1)u/b$, and set $\mathbb{E}(\omega, L, Q, P) := \odot(\omega^{-1} e^{L+Q} \omega (1 + \epsilon \omega^{-4} P) : (i: v_i c_i w_i))$. Now $\omega \in R_S := \mathbb{Z}[t_i, t_i^{-1}]$ is Laurent, $L = \sum l_{ij} \log(t_i) c_j$ with $l_{ij} \in \mathbb{Z}$, $Q = \sum q_{ij} v_i w_j$ with $q_{ij} \in R_S$, and P is a quartic polynomial in v_i, c_j, w_k with coefficients in R_S . The operations are lightly modified, and the $\Lambda\acute{o}\gamma\omicron\varsigma$ and the values of the generators become somewhat simpler, as in the implementation below.

Rough complexity estimate, after $t_k \rightarrow t$: n : xing $\frac{n}{A} \sum_{d=0}^4 \frac{w^{4-d} w^d n^2}{E F G} = n^3 w^4 \in [n^5, n^7]$ number; w : width, maybe $\sim \sqrt{n}$. A : go over stitchings in order. B : multiplication ops per $N^{u_i w_j}$. d : deg of u_i, w_j in P . E : #terms of deg d in P . F : ops per term. G : cost per polynomial multiplication op.

Experimental Analysis ($\omega\epsilon\beta/\text{Exp}$). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Conjecture (checked on the same collections). Given a knot K with Alexander polynomial A , there is a polynomial ρ_1 such that

$$P = A^2 \frac{(t-1)^3 \rho_1 + t^2 (2vw + (1-t)(1-2c)) AA'}{(1-t)t}$$

Furthermore, A and ρ_1 are symmetric under $t \rightarrow t^{-1}$, so let A^+ and ρ_1^+ be their “positive parts”, so e.g., $\rho_1(t) = \rho_1^+(t) + \rho_1^+(t^{-1}) - \rho_1^+(0)$.

Power. On the 250 knots with at most 10 crossings, the pair (A, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always $\deg \rho_1^+ \leq 2g - 1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.