

The Dogma is Wrong

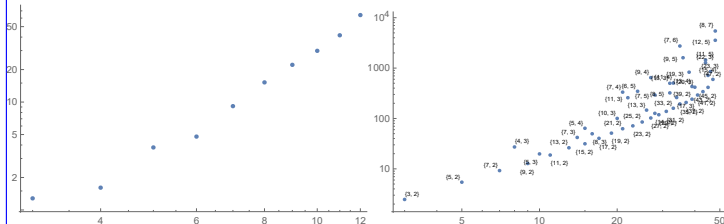


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Abstract. It has long been known that there are knot invariants associated to semi-simple Lie algebras, and there has long been a dogma as for how to extract them: “quantize and use **representation theory**”. We present an alternative and better procedure: “centrally extend, **approximate by solvable**, and learn how to **re-order exponentials** in a universal enveloping algebra”. While equivalent to the old invariants via a complicated process, our invariants are in practice stronger, faster to compute (poly-time vs. exp-time), and clearly carry topological information.

KiW 43 Abstract (wεβ/kiw). Whether or not you like the formulas on this page, they describe the strongest truly computable knot invariant we know.

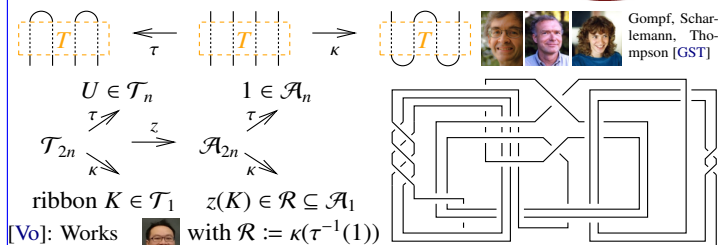
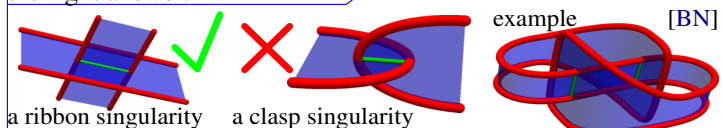
Experimental Analysis (wεβ/Exp). Log-log plots of computation time (sec) vs. crossing number, for all knots with up to 12 crossings (mean times) and for all torus knots with up to 48 crossings:



Power. On the 250 knots with at most 10 crossings, the pair (ω, ρ_1) attains 250 distinct values, while (Khovanov, HOMFLY-PT) attains only 249 distinct values. To 11 crossings the numbers are (802, 788, 772) and to 12 they are (2978, 2883, 2786).

Genus. Up to 12 xings, always ρ_1 is symmetric under $t \leftrightarrow t^{-1}$. With ρ_1^+ denoting the positive-degree part of ρ_1 , always $\deg \rho_1^+ \leq 2g - 1$, where g is the 3-genus of K (equality for 2530 knots). This gives a lower bound on g in terms of ρ_1 (conjectural, but undoubtedly true). This bound is often weaker than the Alexander bound, yet for 10 of the 12-xing Alexander failures it does give the right answer.

Ribbon Knots.



[Vo]: Works for Alexander!



with $\mathcal{R} := \kappa(\tau^{-1}(1))$

$$A^+ = -t^8 + 2t^7 - t^6 - 2t^4 + 5t^3 - 2t^2 - 7t + 13$$

$$\rho_1^+ = 5t^{15} - 18t^{14} + 33t^{13} - 32t^{12} + 2t^{11} + 42t^{10} - 62t^9 - 8t^8 + 166t^7 - 242t^6 + 108t^5 + 132t^4 - 226t^3 + 148t^2 - 11t - 36$$

Ordering Symbols. $\odot$ (*poly* | *specs*) plants the variables of *poly* in $S(\otimes_{ij} g)$ on several tensor copies of $\mathcal{U}(g)$ according to *specs*. E.g.,

$$\odot(a_1^3 y_1 a_2 e^{y_3} x_3^2 | x_3 a_1 \otimes y_1 y_3 a_2) = x^9 a^3 \otimes y e^y a \in \mathcal{U}(g) \otimes \mathcal{U}(g)$$

This enables the description of elements of $\hat{\mathcal{U}}(g)^{\otimes S}$ using commutative polynomials / power series.

Theorem ([BNG], conjectured [MM], elucidated [Ro1]). Let $J_d(K)$ be the coloured Jones polynomial of K , in the d -dimensional representation of sl_2 . Writing

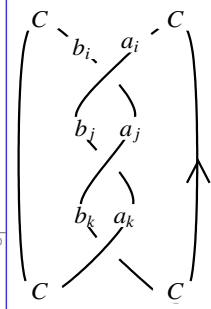
$$\frac{(q^{1/2} - q^{-1/2})J_d(K)}{q^{d/2} - q^{-d/2}} \Big|_{q=e^h} = \sum_{j,m \geq 0} a_{jm}(K) d^j h^m,$$

“below diagonal” coefficients vanish, $a_{jm}(K) = 0$ if $j > m$, and “on diagonal” coefficients give the inverse of the Alexander polynomial:

$$\left(\sum_{m=0}^{\infty} a_{mm}(K) h^m \right) \cdot \omega(K)(e^h) = 1.$$

“Above diagonal” we have **Rozansky’s Theorem** [Ro3, (1.2)]:

$$J_d(K)(q) = \frac{q^d - q^{-d}}{(q - q^{-1})\omega(K)(q^d)} \left(1 + \sum_{k=1}^{\infty} \frac{(q-1)^k \rho_k(K)(q^d)}{\omega^{2k}(K)(q^d)} \right).$$



The Yang-Baxter Technique. Given an algebra U (typically $\hat{\mathcal{U}}(g)$ or $\hat{\mathcal{U}}_q(g)$) and elements

$$R = \sum a_i \otimes b_i \in U \otimes U \quad \text{and} \quad C \in U,$$

form

$$Z = \sum_{i,j,k} C a_i b_j a_k C^2 b_i a_j b_k C.$$

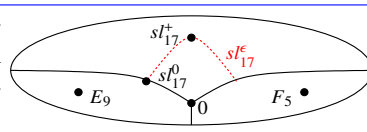
Problem. Extract information from Z .

The Dogma. Use representation theory. In principle finite, but *slow*.

The Loyal Opposition. For certain algebras, work in a homomorphic poly-dimensional “space of formulas”.

$$m_k^{ij} \hookrightarrow \{\mathcal{F}_S\} \xrightarrow{\mathbb{E}} \{U^{\otimes S}\} \xleftarrow{m_k^{ij}}$$

The (fake) moduli of Lie algebras on V , a quadratic variety in $(V^*)^{\otimes 2} \otimes V$ is on the right. We care about $sl_{17}^k := sl_{17}^k / (\epsilon^{k+1} = 0)$.



Recomposing gl_n . Half is enough! $gl_n \oplus \mathfrak{a}_n = \mathcal{D}(\nabla, b, \delta)$:

$$b(\nabla) = b: \nabla \otimes \nabla \rightarrow \nabla$$

$$b(\delta) \sim \delta: \nabla \rightarrow \nabla \otimes \nabla$$

Now define $gl_n^\epsilon := \mathcal{D}(\nabla, b, \epsilon\delta)$. Schematically, this is $[\nabla, \nabla] = \nabla$, $[\delta, \delta] = \epsilon\delta$, and $[\nabla, \delta] = \delta + \epsilon\nabla$. In detail, it is

$$\begin{aligned} [x_{ij}, x_{kl}] &= \delta_{jk} x_{il} - \delta_{li} x_{kj} & [y_{ij}, y_{kl}] &= \epsilon \delta_{jk} y_{il} - \epsilon \delta_{li} y_{kj} \\ [x_{ij}, y_{kl}] &= \delta_{jk} (\epsilon \delta_{i<j} x_{il} + \delta_{il} (b_i + \epsilon a_i) / 2 + \delta_{i>j} y_{il}) \\ &\quad - \delta_{li} (\epsilon \delta_{k<j} x_{kj} + \delta_{kj} (b_j + \epsilon a_j) / 2 + \delta_{k>j} y_{kj}) \\ [a_i, x_{jk}] &= (\delta_{ij} - \delta_{ik}) x_{jk} & [b_i, x_{jk}] &= \epsilon (\delta_{ij} - \delta_{ik}) x_{jk} \\ [a_i, y_{jk}] &= (\delta_{ij} - \delta_{ik}) y_{jk} & [b_i, y_{jk}] &= \epsilon (\delta_{ij} - \delta_{ik}) y_{jk} \end{aligned}$$

The Main sl_2 Theorem. Let $g^\epsilon = \langle t, y, a, x \rangle / ([t, \cdot] = 0, [a, x] = x, [a, y] = -y, [x, y] = t - 2\epsilon a)$ and let $g_k = g^\epsilon / (\epsilon^{k+1} = 0)$. The g_k -invariant of any S -component tangle K can be written in the form $Z(K) = \odot(\omega \epsilon^{L+Q+P}: \bigotimes_{i \in S} y_i a_i x_i)$, where ω is a scalar (a rational function in the variables t_i and their exponentials $T_i := e^{t_i}$), where $L = \sum l_{ij} t_i a_j$ is a quadratic in t_i and a_j with integer coefficients l_{ij} , where $Q = \sum q_{ij} y_i x_j$ is a quadratic in the variables y_i and x_j with scalar coefficients q_{ij} , and where P is a polynomial in $\{\epsilon, y_i, a_i, x_i\}$ (with scalar coefficients) whose ϵ^d -term is of degree at most $2d + 2$ in $\{y_i, \sqrt{a_i}, x_i\}$. Furthermore, after setting $t_i = t$ and $T_i = T$ for all i , the invariant $Z(K)$ is poly-time computable.