

Work in Progress!

The Brute and the Hidden Paradise

Local Algebra (with van der Veen) Much can be reformulated as (non-standard) “quantum algebra” for the 4D Lie algebra $\mathfrak{g} = \langle b, c, u, w \rangle$ over $\mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and $[w, c] = w$, $[c, u] = u$, and $[u, w] = b - 2\epsilon c$. The key: $a_{ij} = (b_i - \epsilon c_i)c_j + u_i w_j$ in $\mathcal{U}(\mathfrak{g})^{\otimes(i,j)}$.



van der Veen

Some (new) representations of the (v-)braid groups.

oeβ/Reps

$B_{i,j}[\xi] := \xi / . v_j \mapsto (1-t) v_i + t v_j$

Burau (old)

Column@{lhs = {v1, v2, v3} // B1,2 // B1,3 // B2,3, rhs = {v1, v2, v3} // B2,3 // B1,3 // B1,2, lhs - rhs // Expand}

... testing R3

{v1, (1-t) v1 + t v2, (1-t) v1 + t ((1-t) v2 + t v3)}
{v1, (1-t) v1 + t v2,
(1-t) ((1-t) v1 + t v2) + t ((1-t) v1 + t v3)}
{0, 0, 0}

$G_{i,j}[\xi] := \xi / . v_j \mapsto (1-t_i) v_i + t_i v_j$

Gassner (old)

... Overcrossings Commute (OC):

Column@{lhs = {v1, v2, v3} // G1,2 // G1,3, Expand[lhs - ({v1, v2, v3} // G1,3 // G1,2)]}

... Undercrossings Commute (UC):

Column@{lhs = {v1, v2, v3} // G1,3 // G2,3, rhs = {v1, v2, v3} // G2,3 // G1,3, lhs - rhs // Expand}

{v1, v2, (1-t1) v1 + t1 ((1-t2) v2 + t2 v3)}
{v1, v2, (1-t2) v2 + t2 ((1-t1) v1 + t1 v3)}
{0, 0, v1 - t1 v1 - t2 v1 + t1 t2 v1 - v2 + t1 v2 + t2 v2 - t1 t2 v2}

Gassner Plus (new?)

$GP_{i,j}[\xi] := \text{Expand}[\xi / . \{u_j \mapsto (1-t_i) u_i + t_i u_j, f \cdot v_j \mapsto f(1-t_i) v_i + f t_i v_j + (t_i - 1)(t_i \partial_{t_i} f - t_j \partial_{t_j} f) u_i + f t_i u_i\}]$;

bas = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3};

Short[lhs = bas // GP1,2 // GP1,3 // GP2,3, 2] ... R3 (left)

{f[t1, t2, t3] v1, f[t1, t2, t3] t1 u1 + f[t1, t2, t3] v1 - f[t1, t2, t3] t1 v1 + <<6>> + t1^2 u1 f^{(1,0,0)}[t1, t2, t3], <<1>> + <<19>> + <<1>>, <<1>>, u1 - t1 u1 + t1 u2, u1 - t1 u1 + t1 u2 - t1 t2 u2 + t1 t2 u3}

... R3 (rest)

(bas // GP2,3 // GP1,3 // GP1,2) - lhs

{0, 0, 0, 0, 0, 0}

... OC

(bas // GP1,2 // GP1,3) - (bas // GP1,3 // GP1,2)

{0, 0, 0, 0, 0, 0}

Question. Does Gassner Plus factor through Gassner?

$K\delta_{i,j} := \text{KroneckerDelta}[i, j]$; Turbo-Gassner (new!)

$TG_{i,j}[\xi] := \text{Expand}[\xi / . \{f \cdot v_k \mapsto \text{Plus}[f v_k / . v_j \mapsto (1-t_i) v_i + t_i v_j, (1-t_i^{-1})(t_i \partial_{t_i} f - t_j \partial_{t_j} f) * (u_k / . u_j \mapsto (1-t_i) u_i + t_i u_j) * u_i w_j, K\delta_{k,i} f (u_j - u_i) u_i w_j, u_j \mapsto (1-t_i) u_i + t_i u_j, w_i \mapsto w_i + (1-t_i^{-1}) w_j, w_j \mapsto t_i^{-1} w_j\}]$;

bas = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3, w1, w2, w3};

Satisfies R3...

(bas // TG1,2 // TG1,3) - (bas // TG1,3 // TG1,2) ... OC

{0, -f[t1, t2, t3] u1 u2 w3 + f[t1, t2, t3] t1 u1 u2 w3 + f[t1, t2, t3] u1 u3 w3 - f[t1, t2, t3] t1 u1 u3 w3, -f[t1, t2, t3] u1 u2 w2 + f[t1, t2, t3] t1 u1 u2 w2 + f[t1, t2, t3] u1 u3 w2 - f[t1, t2, t3] t1 u1 u3 w2, 0, 0, 0, 0, 0, 0}

$\eta / : \eta[i_-]^2 = 0; \eta / : \eta[i_-] \eta[j_-] = 0;$

Turbo-Burau (new!)

$TB_{i,j}[\xi] :=$

Expand[$\xi / . \{f \cdot v_k \mapsto \text{Plus}[f v_k / . v_j \mapsto (1-t-\eta[i]) v_i + (t+\eta[i]) v_j, (t-1)(\text{Coefficient}[f, \eta[i]] - \text{Coefficient}[f, \eta[j]]) * (u_k / . u_j \mapsto (1-t) u_i + t u_j) * u_i w_j, K\delta_{k,i} (f / . _ \eta \rightarrow 0) (u_j - u_i) u_i w_j, u_j \mapsto (1-t) u_i + t u_j, w_i \mapsto w_i + (1-t^{-1}) w_j, w_j \mapsto t^{-1} w_j\}]$;

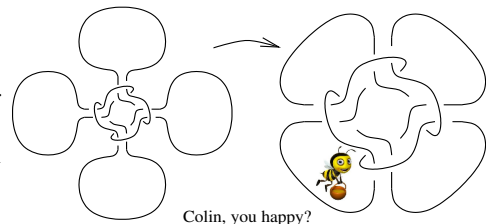
ff = f0 + f1 η[1] + f2 η[2] + f3 η[3];

bas = {ff v1, ff v2, ff v3, u1^2 w1, u2^2 w2, u1, u2, u3, w1, w2, w3};

(bas // TB1,2 // TB1,3) - (bas // TB1,3 // TB1,2) ... OC

{0, -f0 u1 u2 w3 + t f0 u1 u2 w3 + f0 u1 u3 w3 - t f0 u1 u3 w3, -f0 u1 u2 w2 + t f0 u1 u2 w2 + f0 u1 u3 w2 - t f0 u1 u3 w2, 0, 0, 0, 0, 0, 0, 0, 0}

Flower Surgery Theorem. A knot is ribbon iff it is the result of n -petal flower surgery (from thin petals to wide petals) on an n -component unlink, for some n .



Colin, you happy?

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“God created the knots, all else in topology is the work of mortals.”

Leopold Kronecker (modified)

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