Demo Programs for 0-Co. ωεβ/Demo

The R-matrices

```
R⁺_{0,i_,j_} := 𝔼[b_i c_j + b_i^{-1} (e^{b_i} - 1) u_i w_j];
R⁻_{0,i_,j_} := 𝔼[-b_i c_j + b_i^{-1} (e^{-b_i} - 1) u_i w_j];
```

Utilities

```
CF[ω_. 𝔼[Q_]] := Simplify[ω] 𝔼[Simplify[Q]];
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := CF@𝔼[Q1 + Q2];
ω1_. 𝔼[Q1_] ≡ ω2_. 𝔼[Q2_] := Simplify[ω1 == ω2 ∧ Q1 == Q2];
```

Normal Ordering Operators

```
N_{(x:w|u)_i_ c_j_→k_}[ω_. 𝔼[Q_]] := CF[
   ω 𝔼[e^γ α x_k + γ c_k + (Q /. c_j | x_i → 0)] /. {γ → ∂_{c_j} Q, α → ∂_{x_i} Q}];
N_{w_i_ u_j_→k_}[ω_. 𝔼[Q_]] := CF[
   ν ω 𝔼[-b_k ν α β + ν β u_k + ν α w_k + ν δ u_k w_k + (Q /. w_i | u_j → 0)] /.
     ν → (1 + b_k δ)^{-1} /.
     {α → ∂_{w_i} Q /. u_j → 0, β → ∂_{u_j} Q /. w_i → 0, δ → ∂_{w_i,u_j} Q}];
```

Stitching

```
m_{i_,j_→k_}[Z_] := Module[{x, z},
  CF[(Z // N_{w_i u_j→x} // N_{c_i u_x→x} // N_{w_x c_j→x}) /. z_{i|j|x} → z_k]]
```

Some calculations for T_0

```
T0 = R⁺_{0,5,1} R⁺_{0,2,4} R⁻_{0,3,6}
```

$$\mathbb{E}\left[b_5 c_1 + b_2 c_4 - b_3 c_6 + \frac{(-1+e^{b_5})\, u_5 w_1}{b_5} + \frac{(-1+e^{b_2})\, u_2 w_4}{b_2} + \frac{(-1+e^{-b_3})\, u_3 w_6}{b_3}\right]$$

```
T0 // m_{1,2→1} // m_{3,4→3} // m_{3,5→3} // m_{3,6→3}
```

$$\frac{1}{1-(-1+e^{b_1})(-1+e^{b_3})}\,\mathbb{E}\Big[b_3 c_1 + b_1 c_3 - b_3 c_3 + \frac{e^{b_3}(-1+e^{b_1})(-1+e^{b_3})\, u_1 w_1}{(-e^{b_1}-e^{b_3}+e^{b_1+b_3})\, b_1} - \frac{e^{b_1}(-1+e^{b_3})\, u_3 w_1}{(-1+(-1+e^{b_1})(-1+e^{b_3}))\, b_3} - \frac{e^{-b_3}(-1+e^{b_3})\, u_3 w_3}{b_3} - \frac{e^{-b_3}(-1+e^{b_1})(-e^{b_3} b_3 u_1 + e^{b_1}(-1+e^{b_3})\, b_1 u_3)\, w_3}{b_1 (b_3 - (-1+e^{b_1})(-1+e^{b_3})\, b_3)}\Big]$$

Verifying meta-associativity

```
Q0 = 𝔼[Sum[f_i c_i, {i, 3}] + Sum[f_{i,j} u_i w_j, {i, 3}, {j, 3}]]
```

$$\mathbb{E}[c_1 f_1 + c_2 f_2 + c_3 f_3 + u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$$

```
(Q0 // m_{1,2→1} // m_{1,3→1}) ≡ (Q0 // m_{2,3→2} // m_{1,2→1})
True
```

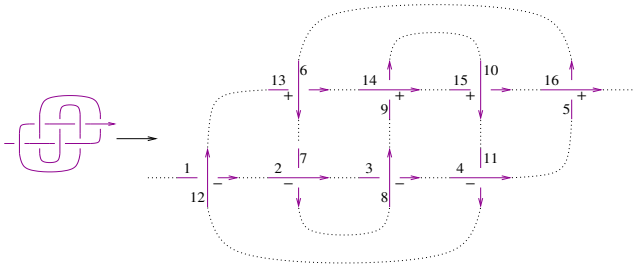
Testing R3

```
t1 = R⁺_{0,1,2} R⁺_{0,3,4} R⁺_{0,5,6} // m_{3,5→x} // m_{1,6→y} // m_{2,4→z}
```

$$\mathbb{E}\left[b_x c_y + b_x c_z + b_y c_z + \frac{e^{b_x}(-1+e^{b_y})\, u_y w_z}{b_y} + \frac{(-1+e^{b_x})\, u_x (w_y + w_z)}{b_x}\right]$$

```
t1 ≡ (R⁺_{0,1,2} R⁺_{0,3,4} R⁺_{0,5,6} // m_{1,3→x} // m_{2,5→y} // m_{4,6→z})
True
```

8_{17}




```
z1 = R⁻_{0,12,1} R⁻_{0,2,7} R⁻_{0,8,3} R⁻_{0,4,11} R⁺_{0,16,5} R⁺_{0,6,13} R⁺_{0,14,9} R⁺_{0,10,15};
Do[z1 = (z1 // m_{1,n→1}) /. b_ → b, {n, 2, 16}];
{CF@z1, KnotData[{8, 17}, "AlexanderPolynomial"][t]}
```

$$\left\{-\frac{e^{3b}\,\mathbb{E}[0]}{1-4e^{b}+8e^{2b}-11e^{3b}+8e^{4b}-4e^{5b}+e^{6b}},\ 11 - \frac{1}{t^3} + \frac{4}{t^2} - \frac{8}{t} - 8t + 4t^2 - t^3\right\}$$

Demo Programs for 1-Co. ωεβ/Demo

The Λόγος

```
Λ[k_] := ((t_k - 1) (2 (α β + δ μ)^2 - α^2 β^2) - 4 v_k c_k w_k δ^2 μ^2 -
    δ (1 + μ) (w_k^2 α^2 + v_k^2 β^2) - v_k^2 w_k^2 δ^3 (1 + 3 μ) -
    2 (α β + 2 δ μ + v_k w_k δ^2 (1 + 2 μ) + 2 c_k δ μ^2) (w_k α + v_k β) -
    4 (c_k μ^2 + v_k w_k δ (1 + μ)) (α β + δ μ)) (1 + t_k) / 4;
```

The Generators

```
R⁺_{i_,j_} := 𝔼[1, Log[t_i] c_j, v_i w_j, v_i c_i w_j + c_i c_j + v_i^2 w_j^2/4];
R⁻_{i_,j_} := 𝔼[1, -Log[t_i] c_j, -t_i^{-1} v_i w_j,
   t_i^{-1} v_i c_j w_j - c_i c_j - t_i^{-2} v_i^2 w_j^2/4];
(ur_{i_} := 𝔼[t_i^{-1/2}, 0, 0, c_i t_i^{-2}]; nr_{i_} := 𝔼[t_i^{1/2}, 0, 0, -c_i t_i^2];)
```

Differential Polynomials

```
DP_{x_→D_α, y_→D_β}[P_][f_] := (* means P[∂_α, ∂_β][f] *)
 Total[CoefficientRules[P, {x, y}] /.
   ({m_, n_} → c_) :→ c D[f, {α, m}, {β, n}]]
```

Utilities

```
CF[ℰ_𝔼] := Expand /@ Together /@ ℰ;
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] 𝔼[ω2_, L2_, Q2_, P2_] :=
  CF@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2^4 P1 + ω1^4 P2];
```

Normal Ordering Operators

```
N_{c_j_ (x:v|w)_i_→k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = e^γ β x_k + γ c_k}, CF[
    𝔼[ω, γ c_k + (L /. c_j → 0), ω e^γ β x_k + (Q /. x_i → 0),
      e^{-q} DP_{c_j→D_γ, x_i→D_β}[P][e^q]] /. {γ → ∂_{c_j} L, β → ω^{-1} ∂_{x_i} Q}]];
N_{w_i_ v_j_→k_}[𝔼[ω_, L_, Q_, P_]] :=
  With[{q = ((1 - t_k) α β + β v_k + α w_k + δ v_k w_k) / μ}, CF[
    𝔼[μ ω, L, μ ω q + μ (Q /. w_i | v_j → 0),
      μ^4 e^{-q} DP_{w_i→D_α, v_j→D_β}[P][e^q] + ω^4 Λ[k]] /. μ → 1 + (t_k - 1) δ /.
     {α → ω^{-1} (∂_{w_i} Q /. v_j → 0), β → ω^{-1} (∂_{v_j} Q /. w_i → 0),
      δ → ω^{-1} ∂_{w_i,v_j} Q}]];
```

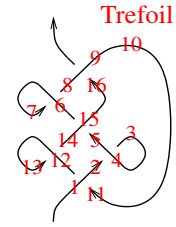
Stitching

```
m_{i_,j_→k_}[Z_𝔼] := Module[{x, z},
  CF[(Z // N_{w_i v_j→x} // N_{c_i v_x→x} // N_{w_x c_j→x}) /. z_{i|j|x} → z_k]]
```

The 0-Framed Trefoil

```
z2 = R⁺_{1,11} R⁻_{4,2} nr_3 R⁺_{15,5} R⁻_{6,8} ur_7 R⁺_{9,16} nr_{10} R⁻_{12,14} ur_{13};
(Do[z2 = z2 // m_{1,k→1}, {k, 2, 16}];
 z2 = z2 /. a_1 :→ a)
```

$$\mathbb{E}\Big[-1 + \frac{1}{t} + t,\ 0,\ 0,\ 16 + \frac{2c}{t^4} - \frac{1}{t^3} - \frac{6c}{t^3} + \frac{4}{t^2} + \frac{10c}{t^2} - \frac{10}{t} - \frac{8c}{t} - 18t + 8ct + 14t^2 - 10ct^2 - 7t^3 + 6ct^3 + 2t^4 - 2ct^4 + 2vw - \frac{2vw}{t^4} + \frac{4vw}{t^3} - \frac{6vw}{t^2} + \frac{2vw}{t} - 6tvw + 4t^2vw - 2t^3vw\Big]$$



Questions and To Do List. • Clean up and write up. • Implement well, compute for everything in sight. • Why are our quantities polynomials rather than just rational functions? • Bounds on their degrees? • Their integrality ($\mathbb{Z}$) properties? • Can everything be re-stated using integrals ($\int$)? • Find the 2-variable version (for knots). How complex is it? • What about links / closed components? • Fully digest the "expansion" theorem; include cuaps. • Explore the (non-)dependence on R. • Is there a canonical R? • What does "group like" mean? • Strand removal? Strand doubling? Strand reversal? • Say something about knot genus. • Find the EK/AT/KV "vertex". • Use as a playground to study associators/braidors. • Restate in topological language. • Study the associated (v-)braid representations. • Study mirror images and the $\mathfrak{b}^+ \leftrightarrow \mathfrak{b}^-$ involution. • Study ribbon knots. • Make precise the relationship with Γ-calculus and Alexander. • Relate to the coloured Jones polynomial. • Relate with "ordinary" q-algebra. • k-smidgen sl_n, etc. • Are there "solvable" CYBE algebras not arising from semi-simple algebras? • Categorify and appease the Gods.