

From  $V$  to  $F$  to  $KV$  following [AT].

```
logF = Λ[V][[1]] // dσ[{x, y} → {y, x}];
logF // EulerE // adSeries[ $\frac{e^{ad}-1}{ad}$ , logF, tb]
```

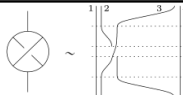
$$\begin{aligned} \overline{x} \rightarrow \text{LS} & \left[ \frac{\overline{y}}{2}, \frac{\overline{xy}}{6}, \frac{1}{24} \overline{xyy}, -\frac{1}{180} \overline{xxxxy} + \frac{1}{80} \overline{xyxy} + \frac{1}{360} \overline{xyyy}, \right. \\ & -\frac{1}{720} \overline{xxxxxy} + \frac{1}{240} \overline{xyxyy} + \frac{1}{240} \overline{xyxyy} + \frac{1}{720} \overline{xyxyxy} - \\ & \frac{\overline{xyyy}}{1440}, \frac{\overline{xxxxxy}}{5040} - \frac{\overline{xxxxy}}{1344} + \frac{13 \overline{xyxyy}}{15120} + \frac{1}{840} \overline{xyxyxy} + \\ & \frac{\overline{xyxyxy}}{3360} + \frac{\overline{xyxyyy}}{6720} + \frac{\overline{xyxyyy}}{1260} + \frac{\overline{xyxyxy}}{1680} - \frac{\overline{xyyy}}{10080}, \dots \Big], \\ \overline{y} \rightarrow \text{LS} & \left[ 0, \frac{\overline{xy}}{12}, \frac{1}{24} \overline{xyy}, -\frac{1}{360} \overline{xxxxy} + \frac{1}{120} \overline{xyxy} + \frac{1}{180} \overline{xyyy}, \right. \\ & -\frac{1}{720} \overline{xxxxxy} + \frac{1}{240} \overline{xyxyy} + \frac{1}{240} \overline{xyxyy} + \frac{1}{720} \overline{xyxyxy} - \\ & \frac{\overline{xyyy}}{1440}, \frac{\overline{xxxxxy}}{10080} - \frac{\overline{xxxxy}}{2016} + \frac{\overline{xyxyy}}{1890} + \frac{\overline{xyxyxy}}{1120} + \frac{\overline{xyxyxy}}{5040} + \\ & \frac{\overline{xyxyyy}}{2520} + \frac{1}{840} \overline{xyxyyy} + \frac{\overline{xyxyxy}}{1260} - \frac{\overline{xyyy}}{5040}, \dots \Big] \end{aligned}$$

```
ϕs[2, 1] = ϕs[3, 1] = ϕs[3, 2] = 0; Solving for an associator Φ.
ϕs[3, 1, 2] = 1/24; ϕ = DKS[3, ϕs];
SeriesSolve[ϕ,
  (ϕσ[3,2,1] ≡ -ϕ) ∧
  (ϕ** ϕσ[1,23,4] ** ϕσ[2,3,4] ≡ ϕσ[12,3,4] ** ϕσ[1,2,34])]];
ϕ (* Can raise degree to 10 *)
```

SeriesSolve::ArbitrarilySetting: In degree 3 arbitrarily setting {ϕs[3, 1, 2] → 0}.  
 SeriesSolve::ArbitrarilySetting: In degree 5 arbitrarily setting {ϕs[3, 1, 1, 1, 2] → 0}.

$$\begin{aligned} \text{DKS} & \left[ 0, \frac{1}{24} \overline{t_{13} t_{23}}, 0, -\frac{7 \overline{t_{13} t_{23} t_{23} t_{23}}}{5760} + \frac{7 \overline{t_{13} t_{13} t_{23} t_{23}}}{5760} - \frac{\overline{t_{13} t_{13} t_{13} t_{23}}}{1440}, \right. \\ & 0, \frac{31 \overline{t_{13} t_{23} t_{23} t_{23} t_{23}}}{967680} - \frac{157 \overline{t_{13} t_{13} t_{23} t_{23} t_{13} t_{23}}}{1935360} - \\ & \frac{31 \overline{t_{13} t_{23} t_{13} t_{23} t_{23} t_{23}}}{387072} - \frac{31 \overline{t_{13} t_{13} t_{23} t_{23} t_{23} t_{23}}}{483840} + \\ & \frac{11 \overline{t_{13} t_{13} t_{13} t_{23} t_{13} t_{23}}}{290304} + \frac{31 \overline{t_{13} t_{13} t_{23} t_{13} t_{23} t_{23}}}{725760} + \frac{83 \overline{t_{13} t_{13} t_{13} t_{23} t_{23} t_{23}}}{967680} - \\ & \left. \frac{13 \overline{t_{13} t_{13} t_{13} t_{13} t_{23} t_{23}}}{241920} + \frac{\overline{t_{13} t_{13} t_{13} t_{13} t_{13} t_{23}}}{60480}, \dots \right] \end{aligned}$$

The "buckle"  $Z_B$ , from  $\Phi$ .



```
R = DKS[t[1, 2] / 2];
Z_B = (-ϕ)σ[13,2,4] ** ϕσ[1,3,2] ** Rσ[2,3] ** (-ϕ)σ[1,2,3] **
  ϕσ[12,3,4];
Z_B@{4}
```

$$\begin{aligned} \text{DKS} & \left[ \frac{\overline{t_{23}}}{2}, -\frac{1}{12} \overline{t_{13} t_{23}} - \frac{1}{24} \overline{t_{14} t_{24}} + \frac{1}{24} \overline{t_{14} t_{34}} + \frac{1}{12} \overline{t_{24} t_{34}}, \right. \\ & 0, \frac{\overline{t_{13} t_{23} t_{23} t_{23}}}{5760} + \frac{7 \overline{t_{14} t_{24} t_{24} t_{24}}}{5760} + \frac{\overline{t_{14} t_{34} t_{24} t_{24}}}{1920} - \\ & \frac{\overline{t_{14} t_{34} t_{34} t_{24}}}{1920} - \frac{7 \overline{t_{14} t_{34} t_{34} t_{34}}}{5760} - \frac{\overline{t_{24} t_{34} t_{34} t_{34}}}{5760} + \frac{\overline{t_{14} t_{24} t_{34} t_{24}}}{1920} + \\ & \frac{\overline{t_{14} t_{24} t_{14} t_{34}}}{1920} - \frac{\overline{t_{14} t_{34} t_{24} t_{34}}}{1920} - \frac{1}{720} \overline{t_{13} t_{13} t_{23} t_{23}} + \\ & \frac{1}{720} \overline{t_{13} t_{13} t_{13} t_{23}} - \frac{7 \overline{t_{14} t_{14} t_{24} t_{24}}}{5760} + \frac{7 \overline{t_{14} t_{14} t_{34} t_{34}}}{5760} - \\ & \frac{\overline{t_{14} t_{24} t_{34} t_{34}}}{5760} + \frac{\overline{t_{14} t_{14} t_{14} t_{24}}}{1440} - \frac{\overline{t_{14} t_{14} t_{14} t_{34}}}{1440} - \frac{1}{960} \overline{t_{14} t_{14} t_{24} t_{34}} + \\ & \left. \frac{\overline{t_{14} t_{24} t_{24} t_{34}}}{5760} - \frac{1}{960} \overline{t_{24} t_{24} t_{34} t_{34}} - \frac{\overline{t_{24} t_{24} t_{24} t_{34}}}{5760}, \dots \right] \end{aligned}$$

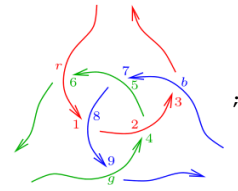
$V$  from  $Z_B$ , following [AET, BND].

```
(E1[Z_B // αMap[1, 2, 3, 4], CWS[0]] // r // τη1 // τη3 //
  ηη2 // ηη4 // hσ[{3} → {2}] // τσ[{2, 4} → {1, 2}])[[
  1]]
```

$$\begin{aligned} 1 \rightarrow \text{LS} & \left[ 0, -\frac{\overline{12}}{24}, 0, \frac{\overline{71112}}{5760} - \frac{\overline{71122}}{5760} + \frac{\overline{1222}}{1440}, 0, \right. \\ & -\frac{31 \overline{111112}}{967680} + \frac{31 \overline{111122}}{483840} - \frac{83 \overline{111222}}{967680} - \frac{31 \overline{112122}}{725760} - \frac{31 \overline{111212}}{645120} + \\ & \frac{13 \overline{112222}}{241920} + \frac{101 \overline{121222}}{1451520} + \frac{527 \overline{112212}}{5806080} - \frac{\overline{122222}}{60480}, \dots \Big], \\ 2 \rightarrow \text{LS} & \left[ \frac{\overline{1}}{2}, -\frac{\overline{12}}{12}, 0, \frac{\overline{1112}}{5760} - \frac{1}{720} \overline{1122} + \frac{1}{720} \overline{1222}, \right. \\ & -\frac{\overline{11112}}{7680} + \frac{\overline{11122}}{3840} - \frac{\overline{11212}}{6912}, \\ & -\frac{\overline{111112}}{645120} + \frac{23 \overline{111122}}{483840} - \frac{13 \overline{111222}}{161280} - \frac{\overline{112122}}{22680} - \frac{41 \overline{111212}}{580608} + \\ & \left. \frac{\overline{112222}}{15120} + \frac{\overline{121222}}{12096} + \frac{71 \overline{112212}}{483840} - \frac{\overline{122222}}{30240}, \dots \right] \end{aligned}$$

The Borromean tangle.

```
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
iRs[a_, b_] := Es[⟨a → LS[0], b → -LS[LW@a]⟩, CWS[0]];
ξ = iRs[r, 6] Rs[2, 4] iRs[g, 9] Rs[5, 7] iRs[b, 3] Rs[8, 1];
```



```
Do[ξ = ξ // dm[r, k, r], {k, 1, 3}];
Do[ξ = ξ // dm[g, k, g], {k, 4, 6}];
Do[ξ = ξ // dm[b, k, b], {k, 7, 9}];
{ξ[[1]]_@{5}, ξ[[2]]_@{5}} // Print
```

$$\begin{aligned} \text{LS} & \left[ 0, \overline{bg}, \frac{1}{2} \overline{bbg} + \overline{bgr} + \frac{1}{2} \overline{bgg}, \right. \\ & \frac{1}{6} \overline{b bbg} + \frac{1}{2} \overline{b bgr} + \frac{1}{2} \overline{b ggr} + \frac{1}{4} \overline{b bbg} + \frac{1}{2} \overline{b grr} + \frac{1}{6} \overline{b ggg}, \\ & \frac{1}{24} \overline{bb bbg} + \frac{1}{6} \overline{bb bgr} + \frac{1}{4} \overline{bb ggr} + \frac{1}{12} \overline{bb bbg} + \\ & \frac{1}{4} \overline{bb grr} + \frac{1}{6} \overline{bg ggr} + \frac{1}{4} \overline{bg grr} - \overline{bgr}g + \\ & \frac{1}{12} \overline{b bbgg} - 2 \overline{b brrg} + \frac{1}{6} \overline{b grrr} + \frac{1}{2} \overline{b bggr} - \\ & \overline{bg brrg} - \frac{1}{12} \overline{bbg bg} - \frac{1}{2} \overline{bgr grr} + \frac{1}{24} \overline{b ggg gg}, \dots \Big], \\ \text{CWS} & \left[ 0, 0, 2 \overline{bgr}, \overline{bbgr} - \overline{bgbr} + \overline{bggr} - \overline{bgrg} + \overline{bgr}r - \overline{brgr}, \frac{\overline{bbgr}}{3} - \right. \\ & \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} - \frac{3 \overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} - \frac{3 \overline{bbgr}}{2} + \frac{\overline{bbgr}}{3} - \\ & \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2} - \frac{3 \overline{bbgr}}{2} + \frac{\overline{bbgr}}{3} + \frac{\overline{bbgr}}{2} - \frac{\overline{bbgr}}{2} + \frac{\overline{bbgr}}{2}, \dots \Big] \end{aligned}$$

References.

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators*, Annals of Mathematics **175** (2012) 415–463, arXiv:0802.4300.  
 [AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld's associators, braid groups and an explicit solution of the Kashiwara-Vergne equations*, Publications Mathématiques de L'IHÉS, **112-1** (2010) 143–189, arXiv:0903.4067  
 [BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects I-IV*, ωεβ/WKO1, ωεβ/WKO2, ωεβ/WKO3, ωεβ/WKO4, and arXiv:1405.1956, arXiv:1405.1955, arXiv: not.yet×2.

Warning. Fidgety!