



Abstract. I will describe a **computable, non-commutative** invariant of tangles with values in wheels, almost generalize it to some balloons, and then tell you why I care. Spoilers: tangles are you know what, wheels are linear combinations of cyclic words in some alphabet, balloons are 2-knots, and one reason I care is because quantum field theory predicts more than I can actually get (but also less).

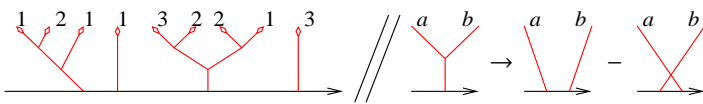
Why I like “non-commutative”? With $FA(x_i)$ the free associative non-commutative algebra,

$$\dim \mathbb{Q}[x, y]_d \sim d \ll 2^d \sim \dim FA(x, y)_d.$$

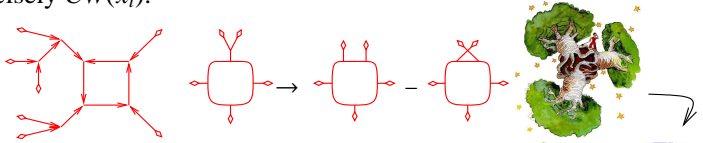
Why I like “computable”?

- Because I’m weird.
- Note that π_1 isn’t computable.

Preliminaries from Algebra. $FL(x_i)$ denotes the free Lie algebra in (x_i) ; $FL(x_i) = (\text{binary trees with AS vertices and coloured leaves}) / (\text{IHX relations})$. There an obvious map $FA(FL(x_i)) \rightarrow FA(x_i)$ defined by $[a, b] \rightarrow ab - ba$, which in itself, is IHX.

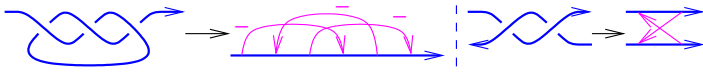


$CW(x_i)$ denotes the vector space of cyclic words in (x_i) : $CW(x_i) = FA(x_i) / (x_i w = w x_i)$. There an obvious map $CW(FL(x_i)) \rightarrow CW(x_i)$. In fact, connected uni-trivalent 2-in-1-out graphs with univalents with colours in $\{1, \dots, n\}$, modulo AS and IHX, is precisely $CW(x_i)$:

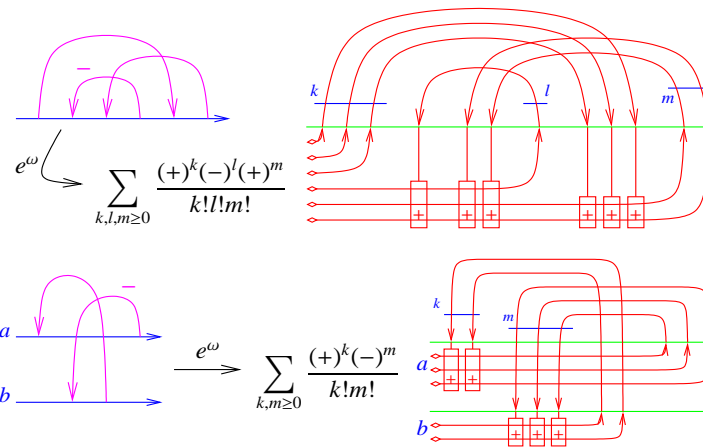


Most important. $e^x = \sum \frac{x^d}{d!}$ and $e^{x+y} = e^x e^y$.

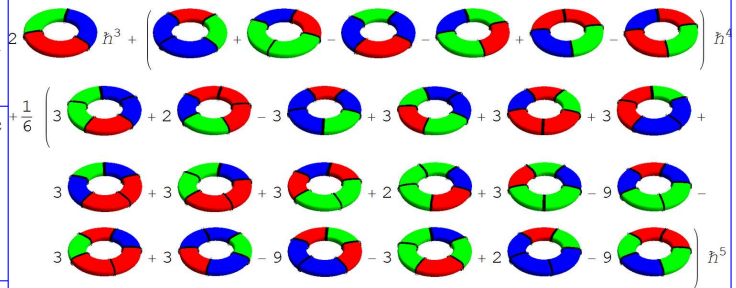
Preliminaries from Knot Theory.



Theorem. ω , the connected part of the procedure below, is an invariant of S -component tangles with values in $CW(S)$:



ω is practically computable! For the Borromean tangle, to degree 5, the result is: (see [BN])

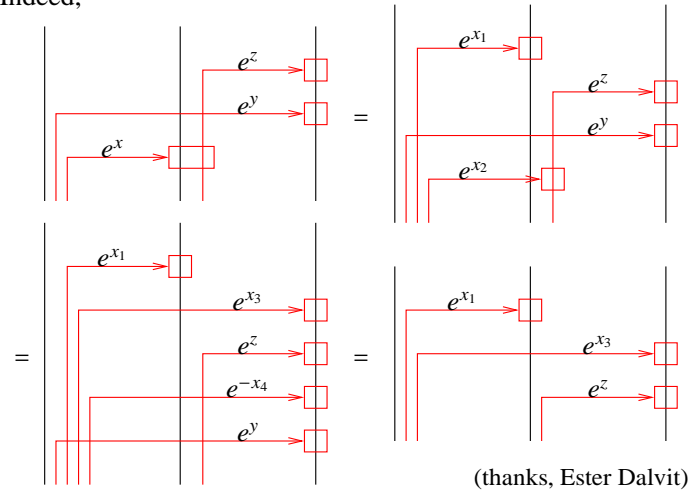


Proof of Invariance.

Need to show:

$$\omega \left(\begin{array}{c} \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \end{array} \right) = \omega \left(\begin{array}{c} \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \end{array} \right)$$

Indeed,

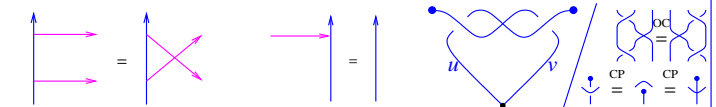


(thanks, Ester Dalvit)

- ω is really the second part of a (trees,wheels)-valued **Further Facts** invariant $\zeta = (\lambda, \omega)$. The tree part λ is just a repackaging of the Milnor μ -invariants.
- On u-tangles, ζ is equivalent to the trees&wheels part of the Kontsevich integral, except it is computable and is defined with no need for a choice of parenthesization.
- On long/round u-knots, ω is equivalent to the Alexander polynomial.
- The multivariable Alexander polynomial (and Levine’s factorization thereof [Le]) is contained in the Abelianization of ζ [BNS].
- ω vanishes on braids.
- Related to / extends Farber’s [Fa]?
- Should be summed and categorified.
- Extends to v and descends to w: meaning, ζ satisfies ω also satisfies so ω ’s “true domain” is



Does ω extend to balloons?



- Agrees with BN-Dancso [BND1, BND2] and with [BN].
- ζ, ω are universal finite type invariants.
- Using $\mathfrak{XK}: v\mathcal{K}_n \rightarrow w\mathcal{K}_{n+1}$, defines a strong invariant of v-tangles / long v-knots. (\mathfrak{XK} in \LaTeX : $\omega \in \beta / \text{zhe}$)