Loading, initializing variables, setting default degree to 6.

(The *Mathematica* packages `FreeLie‘` and `AwCalculus‘` are at ωεβ/WKO4).

```
path = "C:/drorbn/AcademicPensieve/";
SetDirectory[path <> "2015-08/LesDiablerets-1508"];
Get[path <> "Projects/WKO4/FreeLie.m"];
Get[path <> "Projects/WKO4/AwCalculus.m"];
x = LW@"x"; y = LW@"y"; u = LW@"u";
$SeriesShowDegree = 6;
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords,
  AllLyndonWords, AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence,
  BracketForm, BS, CC, Crop, cw, CW, CWS, CWSeries, D, Deg, DegreeScale,
  DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, Inverse, j, J, JA,
  LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism,
  New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t,
  tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌐, ¬}.

FreeLie` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends
 {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA, hm, hS, hΔ, hη,
  hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed to
  support it within reason until July 15, 2022. This is version 150814.
```

```
BCH[x, y] (* Can raise degree to 22 *)
```

$$\mathrm{LS}\Big[\overline{x}+\overline{y},\ \frac{\overline{xy}}{2},\ \frac{1}{12}\overline{x\overline{xy}}+\frac{1}{12}\overline{\overline{xy}y},\ \frac{1}{24}\overline{x\overline{\overline{xy}y}},\ -\frac{1}{720}\overline{x\overline{x\overline{x\overline{xy}}}}+\frac{1}{180}\overline{x\overline{x\overline{\overline{xy}y}}}+\frac{1}{180}\overline{x\overline{\overline{\overline{xy}y}y}}+\frac{1}{120}\overline{\overline{xy}\overline{\overline{xy}y}}+\frac{1}{360}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{1}{720}\overline{\overline{\overline{\overline{xy}y}y}y},\ -\frac{\overline{\overline{\overline{xxx}\overline{xy}}}}{1440}+\frac{1}{360}\overline{x\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}+\frac{1}{240}\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}+\frac{1}{720}\overline{x\overline{x\overline{\overline{xy}\overline{xy}}}}-\frac{\overline{x\overline{\overline{\overline{\overline{\overline{xy}y}y}y}y}}}{1440},\ \ldots\Big]$$

KV Direct.

```
{F = LS[{x, y}, Fs], G = LS[{x, y}, Gs]}; Fs["y"] = 1/2;
SeriesSolve[{F, G},
  ℏ^-1 (LS[x + y] - BCH[y, x] ≡ F - G - Ad[-x][F] + Ad[y][G]) ⋀
   div_x[F] + div_y[G] ≡
    1/2 tr_u[adSeries[ad/(e^ad - 1), x][u] + adSeries[ad/(e^ad - 1), y][u] -
      adSeries[ad/(e^ad - 1), BCH[y, x]][u]]];
{F, G} (* Can raise degree to 13 *)
```

$$\Big\{\mathrm{LS}\Big[\frac{\overline{y}}{2},\ \frac{\overline{xy}}{6},\ \frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{180}\overline{x\overline{x\overline{xy}}}+\frac{1}{80}\overline{x\overline{\overline{xy}y}}+\frac{1}{360}\overline{\overline{\overline{xy}y}y},\ -\frac{1}{720}\overline{x\overline{x\overline{\overline{xy}y}}}+\frac{1}{240}\overline{x\overline{\overline{\overline{xy}y}y}}+\frac{1}{240}\overline{\overline{xy}\overline{\overline{xy}y}}+\frac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{\overline{\overline{\overline{\overline{xy}y}y}y}}{1440},\ \frac{\overline{\overline{\overline{xxx}\overline{xy}}}}{5040}-\frac{\overline{\overline{\overline{xxx}\overline{xy}}}}{1344}+\frac{13\overline{\overline{xx\overline{\overline{xy}y}}}}{15120}+\frac{1}{840}\overline{x\overline{\overline{x\overline{\overline{xy}y}}}}+\frac{\overline{\overline{x\overline{\overline{x\overline{xy}}}}\overline{xy}}}{3360}+\frac{\overline{\overline{x\overline{\overline{\overline{xy}y}y}}y}}{6720}+\frac{\overline{\overline{\overline{xy}\overline{\overline{xy}y}}y}}{1260}+\frac{\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}{1680}-\frac{\overline{\overline{\overline{\overline{xy}y}y}y}}{10080},\ \ldots\Big],$$

$$\mathrm{LS}\Big[0,\ \frac{\overline{xy}}{12},\ \frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{360}\overline{x\overline{x\overline{xy}}}+\frac{1}{120}\overline{x\overline{\overline{xy}y}}+\frac{1}{180}\overline{\overline{\overline{xy}y}y},\ -\frac{1}{720}\overline{x\overline{x\overline{\overline{xy}y}}}+\frac{1}{240}\overline{x\overline{\overline{\overline{xy}y}y}}+\frac{1}{240}\overline{\overline{xy}\overline{\overline{xy}y}}+\frac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{\overline{\overline{\overline{\overline{xy}y}y}y}}{1440},\ \frac{\overline{\overline{\overline{xxx}\overline{xy}}}}{10080}-\frac{\overline{\overline{\overline{xxx}\overline{xy}}}}{2016}+\frac{\overline{\overline{xx\overline{\overline{xy}y}}}}{1890}+\frac{\overline{\overline{x\overline{xy}}\overline{xy}}}{1120}+\frac{\overline{x\overline{x\overline{xy}}\overline{xy}}}{5040}+\frac{\overline{x\overline{\overline{\overline{xy}y}y}y}}{2520}+\frac{1}{840}\overline{\overline{xy}\overline{\overline{\overline{\overline{xy}y}y}y}}+\frac{\overline{\overline{x\overline{\overline{xy}y}}\overline{xy}}}{1260}-\frac{\overline{\overline{\overline{\overline{xy}y}y}y}}{5040},\ \ldots\Big]\Big\}$$

Meaningless calculations.

```
{b[F, G], tr_x[F]}
```

$$\Big\{\mathrm{LS}\Big[0,\ 0,\ -\frac{1}{24}\overline{\overline{xy}y},\ -\frac{1}{48}\overline{\overline{\overline{xy}y}y},\ \frac{1}{720}\overline{x\overline{x\overline{\overline{xy}y}}}-\frac{1}{240}\overline{x\overline{\overline{\overline{xy}y}y}}-\frac{\overline{\overline{xy}\overline{\overline{xy}y}}}{1440}-\frac{1}{720}\overline{\overline{x\overline{xy}}\overline{xy}}-\frac{1}{360}\overline{\overline{\overline{\overline{xy}y}y}y},\ \frac{\overline{x\overline{x\overline{\overline{\overline{xy}y}y}}}}{1440}-\frac{1}{480}\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}y}-\frac{1}{288}\overline{\overline{xy}\overline{\overline{\overline{\overline{xy}y}y}y}}-\frac{7\overline{x\overline{\overline{\overline{xy}y}\overline{xy}}}}{2880}+\frac{\overline{\overline{\overline{\overline{xy}y}y}yyy}}{2880},\ \ldots\Big],$$

$$\mathrm{CWS}\Big[-\frac{\overline{y}}{6},\ \frac{\overline{yy}}{24},\ \frac{\overline{xxy}}{180}+\frac{\overline{xyy}}{80}-\frac{\overline{yyy}}{360},\ -\frac{\overline{xxyy}}{180}+\frac{\overline{xyxy}}{240}-\frac{\overline{xyyy}}{240}-\frac{\overline{yyyy}}{1440},\ -\frac{\overline{xxxxy}}{5040}+\frac{\overline{xxxyy}}{6720}-\frac{\overline{xxyxy}}{1120}+\frac{2\overline{xxyyy}}{945}-\frac{\overline{xyxyy}}{336}+\frac{\overline{xyyyy}}{6720}+\frac{\overline{yyyyy}}{10080},\ \frac{\overline{xxxyy}}{3360}-\frac{\overline{xxxyxy}}{1344}-\frac{\overline{xxxyyy}}{2240}+\frac{\overline{xxyxxy}}{2016}+\frac{13\overline{xxyxyy}}{10080}+\frac{\overline{xxyyxy}}{1680}-\frac{\overline{xxyyyy}}{3780}-\frac{\overline{xyxyxy}}{840}+\frac{\overline{xyxyyy}}{5040}+\frac{\overline{xyyxyy}}{2240}+\frac{\overline{xyyyyy}}{6720}+\frac{\overline{yyyyyy}}{60480},\ \ldots\Big]\Big\}$$

(Also implemented: ∂_λ and derivations in general, `tb`, e^{∂_λ} and morphisms in general, `div`, `j`, Drinfel'd-Kohno, etc.)

The [BND] "vertex" equations.

```
α = LS[{x, y}, αs]; β = LS[{x, y}, βs];
γ = CWS[{x, y}, γs];
V = Es[⟨x → α, y → β⟩, γ];
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
R4Eqn = V ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V;
UnitarityEqn =
  (V ** (V // dA) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]]);
CapEqn = ((V ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
    (Cap (Cap // dσ[x, y]) // dc[x] // dc[y]));
βs["x"] = 1/2; βs["y"] = 0;
SeriesSolve[{α, β, γ, κ},
  (ℏ^-1 R4Eqn) ⋀ UnitarityEqn ⋀ CapEqn];
{V, κ}
```

```
SeriesSolve::ArbitrarilySetting : In degree 1 arbitrarily setting {κs[x] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {αs[x, y, y] → 0}.
SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {αs[x, x, x, y, y] → 0}.
General::stop :
  Further output of SeriesSolve::ArbitrarilySetting will be suppressed during this calculation. >>
```

$$\Big\{\mathrm{Es}\Big[\Big\langle\overline{x}\to\mathrm{LS}\Big[0,\ -\frac{\overline{xy}}{24},\ 0,\ \frac{7\overline{x\overline{x\overline{xy}}}}{5760}-\frac{7\overline{x\overline{\overline{xy}y}}}{5760}+\frac{\overline{\overline{\overline{xy}y}y}}{1440},\ 0,\ -\frac{31\overline{xx\overline{x\overline{xy}}}}{967680}+\frac{31\overline{xx\overline{\overline{\overline{xy}y}}}}{483840}-\frac{83\overline{xx\overline{\overline{\overline{xy}y}y}}}{967680}-\frac{31\overline{x\overline{xy}\overline{\overline{xy}y}}}{725760}-\frac{31\overline{x\overline{x\overline{xy}}\overline{xy}}}{645120}+\frac{13\overline{x\overline{\overline{\overline{xy}y}yy}}}{241920}+\frac{101\overline{\overline{xy}\overline{\overline{xy}y}y}}{1451520}+\frac{527\overline{x\overline{\overline{xy}y}\overline{xy}}}{5806080}-\frac{\overline{\overline{\overline{xy}y}yyy}}{60480},\ \ldots\Big],$$

$$\overline{y}\to\mathrm{LS}\Big[\frac{\overline{x}}{2},\ -\frac{\overline{xy}}{12},\ 0,\ \frac{\overline{x\overline{x\overline{xy}}}}{5760}-\frac{1}{720}\overline{x\overline{\overline{\overline{xy}y}y}}+\frac{1}{720}\overline{\overline{\overline{\overline{xy}y}y}y},\ -\frac{\overline{xx\overline{x\overline{xy}}}}{7680}+\frac{\overline{xx\overline{\overline{\overline{xy}y}}}}{3840}-\frac{\overline{x\overline{\overline{xy}y}\overline{xy}}}{6912},\ -\frac{\overline{xxx\overline{x\overline{xy}}}}{645120}+\frac{23\overline{xxx\overline{\overline{\overline{xy}y}}}}{483840}-\frac{13\overline{xx\overline{\overline{\overline{xy}y}y}}}{161280}-\frac{\overline{x\overline{\overline{xy}\overline{\overline{xy}y}}}}{22680}-\frac{41\overline{x\overline{x\overline{\overline{xy}y}}\overline{xy}}}{580608}+\frac{\overline{x\overline{\overline{\overline{\overline{xy}y}y}y}}}{15120}+\frac{\overline{\overline{xy}\overline{\overline{\overline{xy}y}y}}}{12096}+\frac{71\overline{x\overline{\overline{xy}y}\overline{xy}}}{483840}-\frac{\overline{\overline{\overline{\overline{xy}y}y}yyy}}{30240},\ \ldots\Big]\Big\rangle,$$

$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xy}}{48},\ 0,\ \frac{\overline{xxxy}}{2880}+\frac{\overline{xxyy}}{2880}+\frac{\overline{xyxy}}{5760}+\frac{\overline{xyyy}}{2880},\ 0,\ -\frac{\overline{xxxxxy}}{120960}-\frac{\overline{xxxxyy}}{120960}-\frac{\overline{xxxyxy}}{120960}-\frac{\overline{xxxyyy}}{120960}-\frac{\overline{xxyxxy}}{241920}-\frac{\overline{xxyxyy}}{120960}-\frac{\overline{xxyyxy}}{120960}-\frac{\overline{xxyyyy}}{120960}-\frac{\overline{xyxyxy}}{362880}-\frac{\overline{xyxyyy}}{120960}-\frac{\overline{xyyxyy}}{120960}-\frac{\overline{xyyyyy}}{120960},\ \ldots\Big]\Big],$$

$$\mathrm{CWS}\Big[0,\ -\frac{\overline{xx}}{96},\ 0,\ \frac{\overline{xxxx}}{11520},\ 0,\ -\frac{\overline{xxxxxx}}{725760},\ \ldots\Big]\Big\}$$

Video and more at http://www.math.toronto.edu/~drorbn/Talks/LesDiablerets-1508/