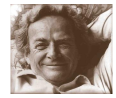
Gaussian Integration, Determinants, Feynman Diagrams

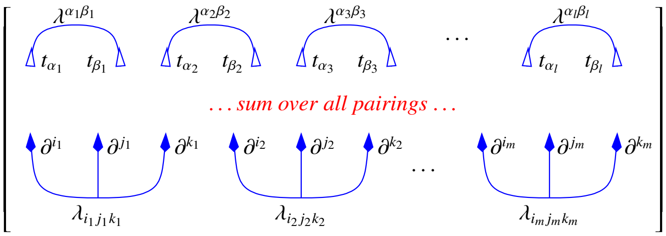
Gaussian Integration. (λ_{ij}) is a symmetric positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

$$\int_{\mathbb{R}^n} e^{-\frac12\lambda_{ij}x^ix^j+\frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k} = \sum_{m\geq 0}\frac{\epsilon^m}{6^m m!}\int_{\mathbb{R}^n}(\lambda_{ijk}x^ix^jx^k)^m e^{-\frac12\lambda_{ij}x^ix^j}$$

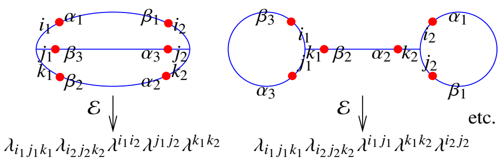

Feynman

$$= \sum_{m\geq 0}\frac{C\epsilon^m}{6^m m!}(\lambda_{ijk}\partial^i\partial^j\partial^k)^m e^{\frac12\lambda^{\alpha\beta}t_\alpha t_\beta}\Big|_{t_\alpha=0} = \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m\left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}$$



$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman diagrams } D}}\mathcal{E}(D)$$



$$= C\sum_{\substack{\text{unmarked Feynman}\\ \text{diagrams } D}}\frac{\epsilon^{m(D)}\mathcal{E}(D)}{|\mathrm{Aut}(D)|}.$$

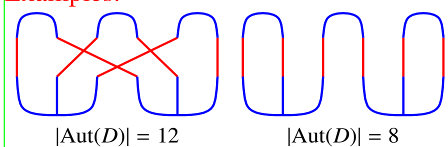
Claim. The number of pairings that produce a given unmarked Feynman diagram D is $\frac{6^m m!2^l l!}{|\mathrm{Aut}(D)|}$.

Proof of the Claim. The group $G_{m,l} := [(S_3)^m \rtimes S_m]\times[(S_2)^l\rtimes S_l]$ acts on the set of pairings, the action is transitive on the set of pairings P that produce a given D, and the stabilizer of any given P is $\mathrm{Aut}(D)$. □

The Fourier Transform. $(F\colon V\to\mathbb{C})\Rightarrow(\tilde f\colon V^*\to\mathbb{C})$ via $\tilde F(\varphi) := \int_V f(v)e^{-i\langle\varphi,v\rangle}dv$. Some facts:

- $\tilde f(0) = \int_V f(v)dv$.
- $\frac{\partial}{\partial\varphi_i}\tilde f \sim \widetilde{v^i f}$.
- $\widetilde{(e^{Q/2})}\sim e^{Q^{-1}/2}$, where Q is quadratic, $Q(v) = \langle Lv,v\rangle$ for $L\colon V\to V^*$, and $Q^{-1}(\varphi) := \langle\varphi, L^{-1}\varphi\rangle$. (This is the key point in the proof of the Fourier inversion formula!)

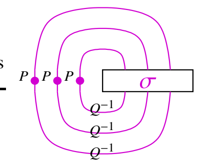
Examples.



Perturbing Determinants. If Q and P are matrices and Q is invertible,

$$|Q|^{-1}|Q+\epsilon P| = |I+\epsilon Q^{-1}P| = \sum_{k\geq 0}\epsilon^k\mathrm{tr}\left(\bigwedge^k Q^{-1}P\right) = \sum_{k\geq 0,\,\sigma\in S_k}\frac{\epsilon^k(-)^\sigma}{k!}\mathrm{tr}\left(\sigma(Q^{-1}P)^{\otimes k}\right)$$

$$= \sum_{k\geq 0,\,\sigma\in S_k}\frac{(-\epsilon)^k(-)^{\#\text{cycles}}}{k!}$$

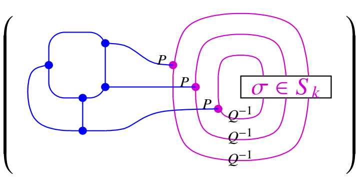


Determinants. Now suppose Q and P_i $(1\leq i\leq n)$ are $d\times d$ matrices and Q is invertible. Then

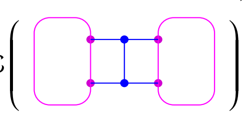
$$|Q|^{-1}I_{\epsilon,\lambda_{ij},\lambda_{ijk},Q,P_i} = |Q|^{-1}\int_{\mathbb{R}^n}e^{-\frac12\lambda_{ij}x^ix^j+\frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k}\det(Q+\epsilon x^iP_i)$$

$$= \sum_{m,k\geq 0,\,\sigma\in S_k}\frac{C\epsilon^{m+k}(-)^\sigma}{6^m m!k!}\int_{\mathbb{R}^n}(\lambda_{ijk}x^ix^jx^k)^m\mathrm{tr}\left(\sigma(x^iQ^{-1}P_i)^{\otimes k}\right)e^{-\frac12\lambda_{ij}x^ix^j}$$

$$= \sum_{\substack{\text{fully marked}\\ \text{Feynman diagrams}}}\frac{C\epsilon^{m+k}(-)^\sigma}{6^m m!k!}\mathcal{E}$$



$$= \sum_{\text{Feynman diagrams}}C\epsilon^{m+k}(-)^k(-)^l\mathcal{E}$$



where l is the number of purple ("Fermion") loops.

Ghosts. Or else, introduce "ghosts" $\bar c_a$ and c^b, write

$$I_{\epsilon,\lambda_{ij},\lambda_{ijk},Q,P_i} = \int_{\mathbb{R}^n}dx\int_{\mathbb{R}^d\times\mathbb{R}^d}e^{-\frac12\lambda_{ij}x^ix^j+\frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k+\bar c_a(Q^a_b+\epsilon x^iP^a_{ib})c^b}$$

and use "ordinary" perturbation theory.


$\bar c$ and c

The Berezin Integral (physics / math language, formulas from `Wikipedia:Grassmann integral`).


Berezin

The *Berezin Integral* is linear on functions of anti-commuting variables, and satisfies $\int\theta d\theta = 1$, and $\int 1d\theta = 0$, so that $\int\frac{\partial f(\theta)}{\partial\theta}d\theta = 0$.

Let V be a vector space, $\theta\in V$, $d\theta\in V^*$ s.t. $\langle d\theta,\theta\rangle = 1$. Then $f\mapsto\int fd\theta$ is the interior multiplication map $\bigwedge V\to\bigwedge V$: $\int fd\theta := i_{d\theta}(f)\left(=\frac{\partial f}{\partial\theta}\right)$.

Multiple integration via "Fubini": $\int f_1(\theta_1)\cdots f_n(\theta_n)d\theta_1\ldots d\theta_n := \left(\int f_1d\theta_1\right)\cdots\left(\int f_nd\theta_n\right)$. $\int fd\theta_1\ldots d\theta_n := f /\!\!/ i_{d\theta_1} /\!\!/ \cdots /\!\!/ i_{d\theta_n}$.

Change of variables. If $\theta_i = \theta_i(\xi_j)$, both θ_i and ξ_j are odd, and $J_{ij} := \partial\theta_i/\partial\xi_j$, then

$$\int f(\theta_i)d\theta = \int f(\theta_i(\xi_j))\det(J_{ij})^{-1}d\xi.$$

Given vector spaces V_{θ_i} and W_{ξ_j}, $d\theta = \bigwedge d\theta_i\in\bigwedge^{\text{top}}(V^*)$, $d\xi = \bigwedge d\xi_i\in\bigwedge^{\text{top}}(W^*)$, and $T\colon V\to\bigwedge^{\text{odd}}(W)$. Then T induces a map $T_*\colon\bigwedge V\to\bigwedge W$ and then

$$\int fd\theta = \int(T_*f)\det\left(\frac{\partial(T\theta_i)}{\partial\xi_j}\right)^{-1}d\xi.$$

Gaussian integration. For an even matrix A and odd vectors θ, η,

$$\int e^{\theta^TA\eta}d\theta d\eta = \det(A),\qquad \int e^{\theta^TA\eta+\theta^TJ+K^T\eta}d\theta d\eta = \det(A)e^{-K^TA^{-1}J}.$$

Video and more at http://drorbn.net/?title=AKT-14 (Feb 28 and March 21 classes)