

What happens to a quantum particle on a pendulum at $T = \frac{\pi}{2}$?

Abstract. This subject is the best one-hour introduction I know for the mathematical techniques that appear in quantum mechanics — in one short lecture we start with a meaningful question, visit Schrödinger’s equation, operators and exponentiation of operators, Fourier analysis, path integrals, the least action principle, and Gaussian integration, and at the end we land with a meaningful and interesting answer.

Based a lecture given by the author in the “trivial notions” seminar in Harvard on April 29, 1989. This edition, January 10, 2014.

1. THE QUESTION

Let the complex valued function $\psi = \psi(t, x)$ be a solution of the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = -i \left(-\frac{1}{2} \Delta_x + \frac{1}{2} x^2 \right) \psi \quad \text{with} \quad \psi|_{t=0} = \psi_0.$$

What is $\psi|_{t=T=\frac{\pi}{2}}$?

In fact, the major part of our discussion will work just as well for the general Schrödinger equation,

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= -iH\psi, & H &= -\frac{1}{2} \Delta_x + V(x), \\ \psi|_{t=0} &= \psi_0, & & \text{arbitrary } T, \end{aligned}$$

where,

- ψ is the “wave function”, with $|\psi(t, x)|^2$ representing the probability of finding our particle at time t in position x .
- H is the “energy”, or the “Hamiltonian”.
- $-\frac{1}{2} \Delta_x$ is the “kinetic energy”.
- $V(x)$ is the “potential energy at x ”.

2. THE SOLUTION

The equation $\frac{\partial \psi}{\partial t} = -iH\psi$ with $\psi|_{t=0} = \psi_0$ formally implies

$$\psi(T, x) = (e^{-iTH} \psi_0)(x) = \left(e^{i\frac{T}{2} \Delta - iTV} \psi_0 \right)(x).$$

By Lemma 3.1 with $n = 10^{58} + 17$ and setting $x_n = x$ we find that $\psi(T, x)$ is

$$\left(e^{i\frac{T}{2n} \Delta} e^{-i\frac{T}{n} V} e^{i\frac{T}{2n} \Delta} e^{-i\frac{T}{n} V} \dots e^{i\frac{T}{2n} \Delta} e^{-i\frac{T}{n} V} \psi_0 \right)(x_n).$$

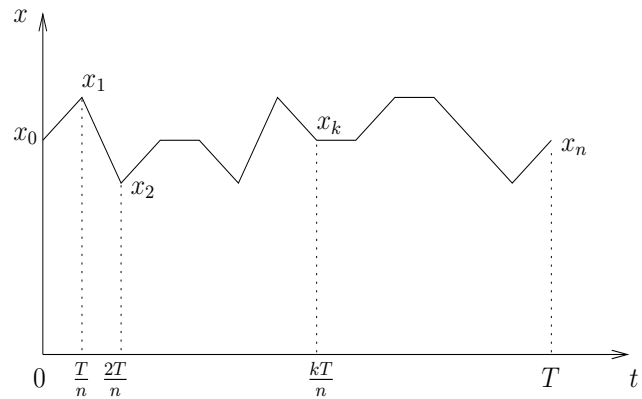
Now using Lemmas 3.2 and 3.3 we find that this is: (c denotes the ever-changing universal fixed numerical constant)

$$\begin{aligned} c \int dx_{n-1} e^{i\frac{(x_n - x_{n-1})^2}{2T/n}} e^{-i\frac{T}{N} V(x_{n-1})} \dots \\ \int dx_1 e^{i\frac{(x_2 - x_1)^2}{2T/n}} e^{-i\frac{T}{N} V(x_1)} \\ \int dx_0 e^{i\frac{(x_1 - x_0)^2}{2T/n}} e^{-i\frac{T}{N} V(x_0)} \psi_0(x_0). \end{aligned}$$

Repackaging, we get

$$\begin{aligned} c \int dx_0 \dots dx_{n-1} \\ \exp \left(i\frac{T}{2n} \sum_{k=1}^n \left(\frac{x_k - x_{k-1}}{T/n} \right)^2 - i\frac{T}{n} \sum_{k=0}^{n-1} V(x_k) \right) \\ \psi_0(x_0). \end{aligned}$$

Now comes the novelty. keeping in mind the picture



and replacing Riemann sums by integrals, we can write

$$\begin{aligned} \psi(T, x) &= c \int dx_0 \int_{W_{x_0 x_n}} \mathcal{D}x \\ &\exp \left(i \int_0^T dt \left(\frac{1}{2} \dot{x}^2(t) - V(x(t)) \right) \right) \psi_0(x_0), \end{aligned}$$

where $W_{x_0 x_n}$ denotes the space of paths that begin at x_0 and end at x_n ,

$$W_{x_0 x_n} = \{x : [0, T] \rightarrow \mathbb{R} : x(0) = x_0, x(T) = x_n\},$$

and $\mathcal{D}x$ is the formal “path integral measure”.

This is a good time to introduce the “action” \mathcal{L} :

$$\mathcal{L}(x) := \int_0^T dt \left(\frac{1}{2} \dot{x}^2(t) - V(x(t)) \right).$$

With this notation,

$$\psi(T, x) = c \int dx_0 \psi_0(x_0) \int_{W_{x_0 x_n}} \mathcal{D}x e^{i\mathcal{L}(x)}.$$