


Example.



T. Kohno

$K = \left\langle \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right\rangle$  (goes back to [Koh])

$I = \left\langle \begin{array}{c} \bullet \\ \diagup \diagdown \\ \diagdown \diagup \end{array} \right\rangle$

$(K/I^{p+1})^* = (\text{invariants of type } p) =: \mathcal{V}_p$

$(I^p/I^{p+1})^* = \mathcal{V}_p/\mathcal{V}_{p-1} \quad V = \langle t^{ij} | t^{ij} = t^{ji} \rangle = \langle \text{HH} \rangle$


$\ker \bar{\mu}_2 = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}] \rangle = \langle 4T \text{ relations} \rangle$

$A = q(K) = \left( \text{horizontal chord diagrams mod } 4T \right) = \langle \text{HH} \rangle / 4T$

Z: universal finite type invariant, the Kontsevich integral.

$PvB_n$  is the group

$\langle \sigma_{ij} : 1 \leq i \neq j \leq n \rangle / \left\langle \begin{array}{l} \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \end{array} \right\rangle$



L. Kauffman [Kau, KL]


of “pure virtual braids” (“braids when you look”, “blunder braids”):

$\sigma_{24} = \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array}$

R3:  $\begin{array}{c} \uparrow^k \quad \uparrow^j \quad \uparrow^i \\ \diagdown \quad \diagup \\ \uparrow^k \quad \uparrow^j \quad \uparrow^i \end{array} = \begin{array}{c} \uparrow^k \quad \uparrow^j \quad \uparrow^i \\ \diagdown \quad \diagup \\ \uparrow^k \quad \uparrow^j \quad \uparrow^i \end{array}$

The Main Theorem [Lee].  $PvB_n$  is quadratic.

$A_n = q(PvB_n)$ .

[GPV]  Goussarov-Polyak-Viro

$I = \left\langle \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array} \right\rangle$  with  $\bowtie = \tilde{\sigma}_{ij} = \sigma_{ij} - 1 = \bowtie - \bowtie$ , the “semi-virtual crossing”.

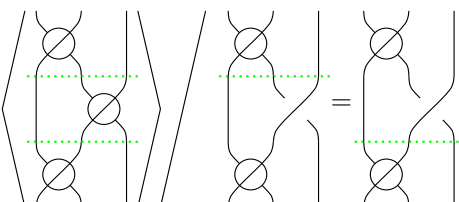
$V = I/I^2 = \left\langle \begin{array}{c} \text{v-braids} \\ \text{with one } \bowtie \end{array} \right\rangle / (\bowtie = \times)$

$A_n = TV / \langle [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}], C_{kl}^{ij} = [a_{ij}, a_{kl}] \rangle$

$y_{ijk} = \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array} + \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array} + \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array} - \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array} - \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array} - \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array}$

$a_{24} = \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array}$

$I^p$

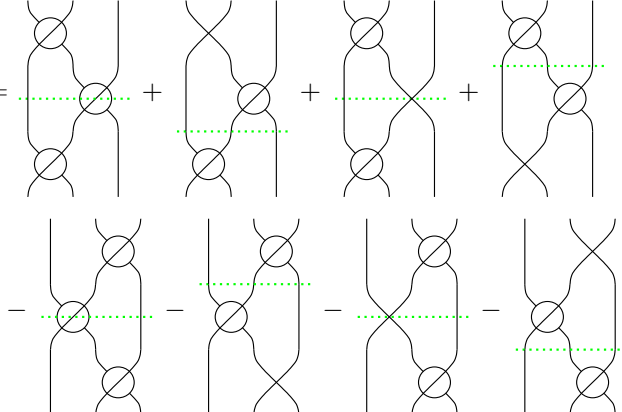


James Gillespie’s Sightline #2 (1984) is a syzygy, and (arguably) Toronto’s largest sculpture. Find it next to University of Toronto’s Hart House.



$\mathfrak{R}_2(PvB_n)$  is generated as a vector space by  $C_{kl}^{ij}$  and

$Y_{ijk} := \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array} + \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array} + \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array} + \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array}$



Syzygy Completeness, for  $PvB_n$ , means:

$$\mathfrak{R}_p = \bigoplus_{j=1}^{p-1} \mathfrak{R}_{p,j} \xrightarrow{\partial} I^p \xrightarrow{\pi} V^{\otimes p}$$

$\{\tilde{\sigma}_{12} : Y_{345} : \tilde{\sigma}_{67} : \dots\} \rightarrow \{\tilde{\sigma}_{12} : Y_{345} : \tilde{\sigma}_{67} : \dots\} \rightarrow \{a_{12}y_{345}a_{67} : \dots\}$

Is every relation between the  $y_{ijk}$ ’s and the  $C_{kl}^{ij}$ ’s also a relation between the  $Y_{ijk}$ ’s and the  $C_{kl}^{ij}$ ’s?

The Group  $PvB_n$

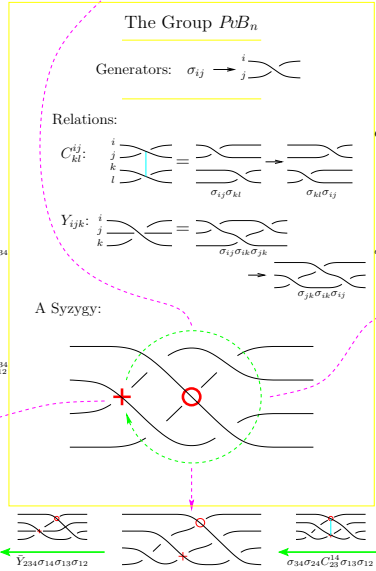
Generators:  $\sigma_{ij} \rightarrow \begin{array}{c} \uparrow \quad \uparrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array}$

Relations:

$C_{kl}^{ij} : \begin{array}{c} \uparrow^i \quad \uparrow^j \\ \diagdown \quad \diagup \\ \uparrow^k \quad \uparrow^l \end{array} = \begin{array}{c} \uparrow^i \quad \uparrow^j \\ \diagdown \quad \diagup \\ \uparrow^k \quad \uparrow^l \end{array} \rightarrow \begin{array}{c} \uparrow^i \quad \uparrow^j \\ \diagdown \quad \diagup \\ \uparrow^k \quad \uparrow^l \end{array}$

$Y_{ijk} : \begin{array}{c} \uparrow^i \quad \uparrow^j \\ \diagdown \quad \diagup \\ \uparrow^k \quad \uparrow^l \end{array} = \begin{array}{c} \uparrow^i \quad \uparrow^j \\ \diagdown \quad \diagup \\ \uparrow^k \quad \uparrow^l \end{array} \rightarrow \begin{array}{c} \uparrow^i \quad \uparrow^j \\ \diagdown \quad \diagup \\ \uparrow^k \quad \uparrow^l \end{array}$

A Syzygy:



Theorem S. Let  $D$  be the free associative algebra generated by symbols  $a_{ij}$ ,  $y_{ijk}$  and  $C_{kl}^{ij}$ , where  $1 \leq i, j, k, l \leq n$  are distinct integers. Let  $D_0$  be the part of  $D$  with only  $a_{ij}$  symbols and let  $D_1$  be the span of the monomials in  $D$  having only  $a_{ij}$  symbols, with exactly one exception that may be either a  $y_{ijk}$  or a  $C_{kl}^{ij}$ . Let  $\partial : D_1 \rightarrow D_0$  be the map defined by

$$y_{ijk} \mapsto [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}],$$

$$C_{kl}^{ij} \mapsto [a_{ij}, a_{kl}].$$

Then  $\ker \partial$  is generated by a family of elements readable from the picture above and by a few similar but lesser families.