

# From the $ax + b$ Lie Algebra to the Alexander Polynomial and Beyond

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http://www.math.toronto.edu/~drorbn/Talks/Chicago-1009/

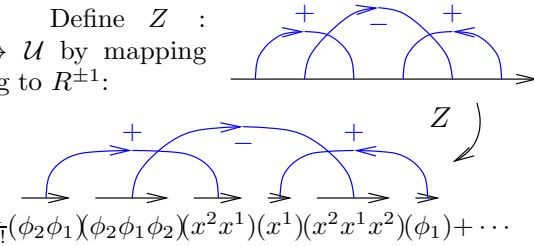
**Abstract.** I will present the simplest-ever “quantum” formula for the Alexander polynomial, using only the unique two dimensional non-commutative Lie algebra (the one associated with the “ $ax + b$ ” Lie group). After introducing the “Euler technique” and some diagrammatic calculus I will sketch the proof of the said formula, and following that, I will present a long list of extensions, generalizations, and dreams.

**The 2D Lie Algebra.** Let  $\mathfrak{g} = \mathfrak{lie}(x^1, x^2)/[x^1, x^2] = x^2$ , let  $\mathfrak{g}^* = \langle \phi_1, \phi_2 \rangle$  with  $\phi_i(x^j) = \delta_i^j$ , let  $I\mathfrak{g} = \mathfrak{g}^* \rtimes \mathfrak{g}$  so  $[\phi_i, \phi_j] = [\phi_i, x^j] = 0$  while  $[x^1, \phi_2] = -\phi_2$  and  $[x^2, \phi_2] = \phi_1$ . Let  $r = Id = \phi_1 \otimes x^1 + \phi_2 \otimes x^2 \in \mathfrak{g}^* \otimes \mathfrak{g} \subset I\mathfrak{g} \otimes I\mathfrak{g}$ . Let  $\mathcal{U} = \{\text{words in } I\mathfrak{g}\}/ab - ba = [a, b]$ , degree-completed with respect to  $\deg \phi_i = 1$  and  $\deg x^i = 0$  (so  $\mathcal{U} \equiv$  (power series is 4 variables)). Let  $R = \exp(r) \in \mathcal{U} \otimes \mathcal{U}$ .

**The Invariant.** Define  $Z : \{\text{long knots}\} \rightarrow \mathcal{U}$  by mapping every  $\pm$ -crossing to  $R^{\pm 1}$ :



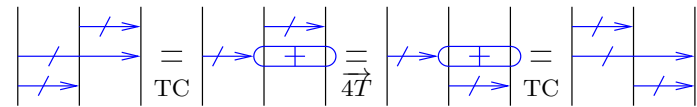
Alexander



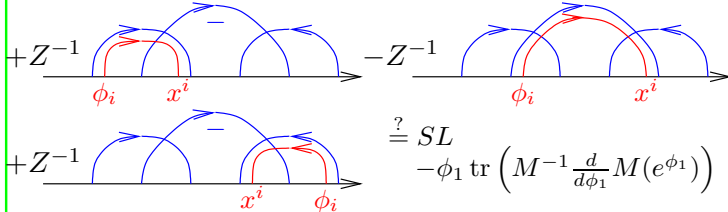
**Near Theorem.**  $Z$  is invariant, and it is essentially the Alexander polynomial; with  $N = \exp(\overleftarrow{t} \phi_i x^i + \overrightarrow{t} x^i \phi_i) =: \exp(SL)$ ,

$$Z(K) = N \cdot (A(K)(e^{\phi_1}))^{-1} \quad (1)$$

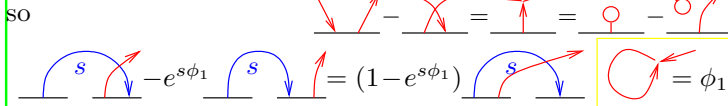
**Invariance.** “The identity is an invariant tensor”:



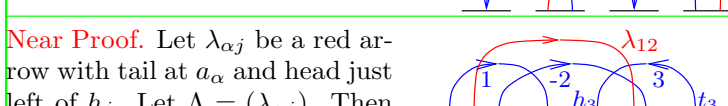
**The Euler Prelude.** Apply  $\tilde{E}\zeta := \zeta^{-1}E\zeta$  to (1):



**Some Relations.**  $\phi_i x^i, x^i \phi_i, \phi_1$  are central,  $x^i \phi_i - \phi_i x^i = \phi_1$ ,  $[x^j, \phi_i] = \delta_i^j \phi_1 - \delta_1^j \phi_i$  or



and the famed “tails commute” (TC):



**Near Proof.** Let  $\lambda_{\alpha_j}$  be a red arrow with tail at  $a_\alpha$  and head just left of  $h_j$ . Let  $\Lambda = (\lambda_{\alpha_j})$ . Then roughly  $R\Lambda = \phi_1 I$  so roughly,  $\Lambda = R^{-1}\phi_1$ . The rest is book-keeping that I haven't finished yet, yet with which my computer agrees fully.

I don't understand the Alexander polynomial!



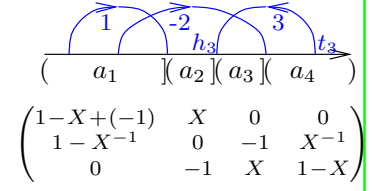
“God created the knots, all else in topology is the work of mortals.”  
Leopold Kronecker (modified)



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## An Alexander Reminder.

Number the arrows  $1, \dots, n$ , let  $t_j, h_j$  be the tail and head of arrow  $j$ , and let  $s_j \in \pm 1$  be its sign. Cut the skeleton into arcs  $a_\alpha$  by arrow heads, and



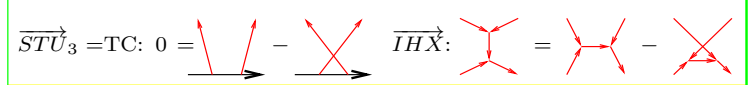
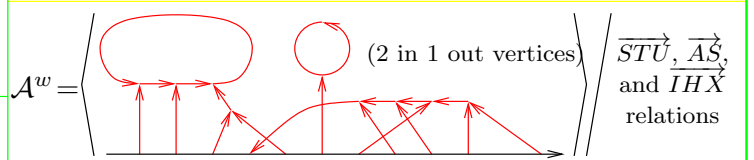
let  $\alpha(p)$  be “the arc of point  $p$ ”. Let  $R \in M_{n \times (n+1)}$  be the matrix whose  $j$ 'th row has  $-1$  in column  $\alpha(h_j)$  and  $1 - X^{s_j}$  in column  $\alpha(t_j)$  and  $X^{s_j}$  in column  $\alpha(h_j) + 1$ , and let  $M$  be  $R$  with a column removed. Then  $A(X) = \det(M)$ .

**An Euler Interlude.** If you know brackets, how do you test exponentials? When's  $e^A e^B = e^C e^D$ ?

**Bad Idea.** Take log and use BCH. You'll want to cry.

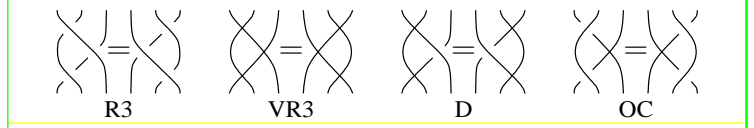
**Clever Idea.** Let  $E$  be the Euler derivation, which multiplies each element by its degree (e.g. on  $\mathbb{Q}[[\phi]]$ ,  $E\phi = \phi \partial_\phi \phi$ , so  $Ee^\phi = \phi e^\phi$ ). Apply  $\tilde{E}\zeta := \zeta^{-1}E\zeta$ :  $\tilde{E}(e^A e^B) = e^{-B} e^{-A} (e^A A e^B + e^A e^B B) = e^{-B} A e^B + B = e^{-\text{ad } B}(A) + B$ .

**“Uninterpreting” Diagrams.** Make  $Z^w : \mathcal{K}^w \rightarrow \mathcal{A}^w \rightarrow \mathcal{U}$ , with

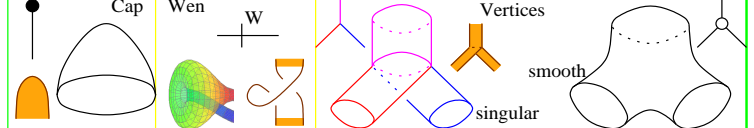
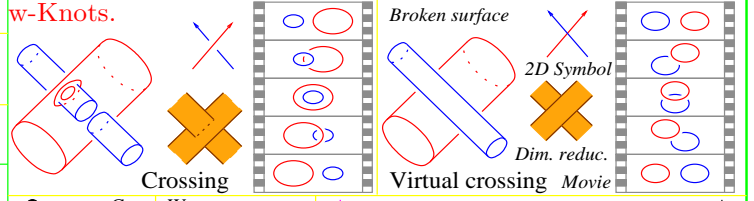


$\mathcal{K}^w = CA \langle \text{crossings} \rangle / \text{R23, OC}$

$= PA \langle \text{crossings} \rangle / \text{R23, VR123, D, OC}$



$Z^w$  is a UFTI on w-knots! It extends to links and tangles, is well behaved under compositions and cables, and remains computable for tangles. It contains Burau, Gassner, and Cimasoni-Turaev in natural ways, and it contains the MVA though my understanding of the latter is incomplete.



There's 1D in 4D, non-trivial given 2D, and there are ops...

**Dream.**  $Z^w$  extends to virtual knots as  $Z^v : \mathcal{K}^v \rightarrow \mathcal{A}^v$ , with good composition and cabling properties and plenty of computable quotients, more than there are quantum groups and representations thereof. I don't understand quantum groups!