1. $\operatorname{proj}\mathcal{K}^w(\uparrow_n) \cong_j \mathcal{U}\left((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n\right)$

— All Signs Are Wrong! —

I understand Drinfel'd and Alekseev–Torossian, I don't understand Etingof–Kazhdan yet, and I'm clueless about Kontsevich

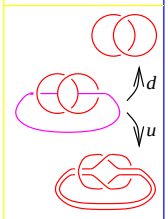
Dror Bar–Natan, Montpellier, June 2010, http://www.math.toronto.edu/~drorbn/Talks/Montpellier–1006/

Cans and Can't Yets.

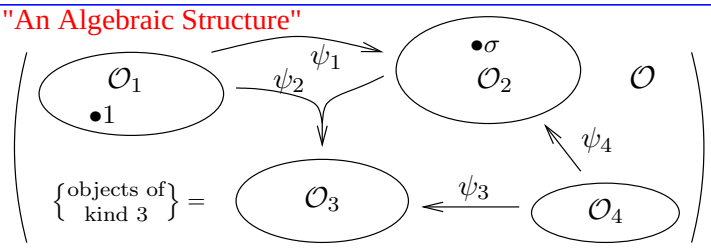
$\left(\begin{matrix}\text{arbitrary algebraic}\\ \text{structure}\end{matrix}\right) \xrightarrow[\text{machine}]{\text{projectivization}} \left(\begin{matrix}\text{a problem in}\\ \text{graded algebra}\end{matrix}\right)$

- Feed knot-things, get Lie algebra things.
- (u-knots)→(Drinfel'd associators).
- (w-knots)→(K-V-A-E-T).
- Dream: (v-knots)→(Etingof-Kazhdan).
- Clueless: (???)→(Kontsevich)?
- Goals: add to the Knot Atlas, produce a working AKT and touch ribbon 1-knots, rip benefits from *truly* understanding quantum groups.

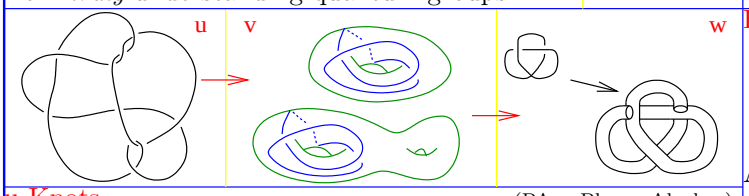
www.katlas.org



"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

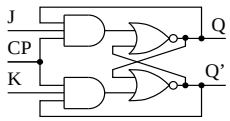


u-Knots

(PA :=Planar Algebra)

$\left\{\begin{matrix}\text{knots}\\ \&\text{links}\end{matrix}\right\} = \text{PA}\left\langle \; \middle| \; \text{R123:} \right\rangle_{0\text{ legs}}$

Circuit Algebras



A J-K Flip Flop

Infineon HYS64T64020HDL-3.7-A 512MB RAM

v-Tangles and w-Tangles

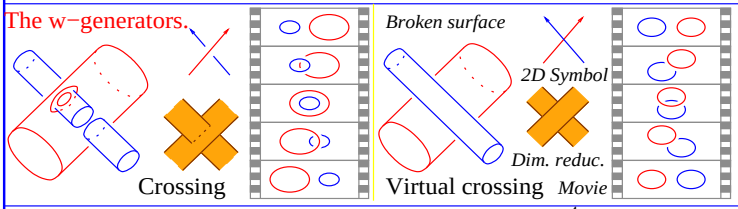
(CA :=Circuit Algebra)

$\left\{\begin{matrix}\text{v-knots}\\ \&\text{ links}\end{matrix}\right\} = \text{CA}\left\langle \; \middle| \; \text{R23:} \right\rangle$

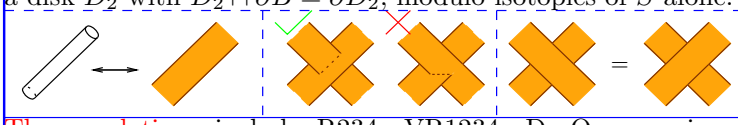
$= \text{PA}\left\langle \; , \; \middle| \; \begin{matrix}\text{VR123:}\\ \text{R23}\end{matrix} \right.$; D: $\Big\rangle$

$\{\text{w-Tangles}\} = \text{v-Tangles} \Big/ \text{OC:}$

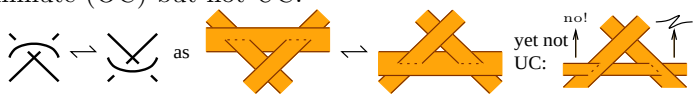
The w–generators.



A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.



The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\begin{array}{ccccccccc} \text{ops}\circlearrowleft\mathcal{K} & = & \mathcal{K}_0 & \supset & \mathcal{K}_1 & \supset & \mathcal{K}_2 & \supset & \mathcal{K}_3 \supset \dots \\ \Downarrow & & & & & & \downarrow Z & & \\ \text{ops}\circlearrowleft \operatorname{gr}\mathcal{K} & := & \mathcal{K}_0/\mathcal{K}_1 & \oplus & \mathcal{K}_1/\mathcal{K}_2 & \oplus & \mathcal{K}_2/\mathcal{K}_3 & \oplus & \mathcal{K}_3/\mathcal{K}_4 \oplus \dots \end{array}$$

An expansion is a filtered $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

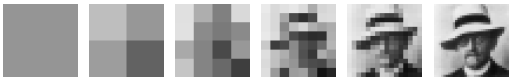
Reality. $\operatorname{gr}\mathcal{K}$ is often too hard. An $\mathcal{A}$-expansion is a graded "guess" $\mathcal{A}$ with a surjection $\tau : \mathcal{A} \to \operatorname{gr}\mathcal{K}$ and a filtered $Z : \mathcal{K} \to \mathcal{A}$ for which $(\operatorname{gr} Z)\circ\tau = I_{\mathcal{A}}$. An $\mathcal{A}$-expansion confirms $\mathcal{A}$ and yields an ordinary expansion. Same for "homomorphic".

Just for fun.

$\mathcal{K} = \left\{ \; \right\} = \left(\begin{matrix}\text{The set of all}\\ \text{b/w 2D projec-}\\ \text{tions of reality}\end{matrix}\right)$

$\mathcal{K}/\mathcal{K}_1 \leftarrow \mathcal{K}/\mathcal{K}_2 \leftarrow \mathcal{K}/\mathcal{K}_3 \leftarrow \mathcal{K}/\mathcal{K}_4 \leftarrow \dots$

Crop Rotate Adjoin



An expansion Z is a choice of a "progressive scan" algorithm.

crop rotate adjoin

$\mathcal{K}/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \dots$

$\| \qquad\qquad \|$

$\mathbb{R} \qquad \ker(\mathcal{K}/\mathcal{K}_4 \to \mathcal{K}/\mathcal{K}_3)$

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1 = \mathcal{I}$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle(\mathcal{K}_1)^m\rangle$ (using all available "products"). In this case, set $\operatorname{proj}\mathcal{K} := \operatorname{gr}\mathcal{K}$.

Examples. 1. The projectivization of a group is a graded associative algebra.

2. Pure braids — PB_n is generated by x_{ij}, "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \operatorname{gr} PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel'd theory of associators.

3. Quandle: a set Q with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

$\operatorname{proj} Q$ is a graded Leibniz algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate I!), feed $1 + \bar{x}$, $1 + \bar{y}$, $1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

Kashiwara, Vergne, Alekseev, Enriquez, Torossian.

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Montpellier-1006/