

# Projectivization, w-Knots, Kashiwara-Vergne and Alekseev-Torossian

"An Algebraic Structure"

## The Categorification Speculative Paradigm.

- Every object in math is the Euler characteristic of a complex.
- Every operation in math lifts to an operation between complexes.
- Every identity in mathematics is true up to homotopy.

## The Projectivization Tentative Speculative Paradigm. Projectivization?

- Every graded algebraic structure in mathematics is the projectivization of a plain ("global") one.
- Every equation written in a graded algebraic structure is an equation for a homomorphic expansion, or for an automorphism of such.

## Graded Equations Examples

- $e(x + y) = e(x)e(y)$  in  $\mathbb{Q}[[x, y]]$ .
- The pentagon and hexagons in  $\mathcal{A}(\uparrow_{3,4})$ .
- The equations defining a QUEA, the work of Etingof and Kazhdan.

## The Alekseev-Torossian equations in $\mathcal{U}(\text{sder}_n)$ and $\mathcal{U}(\text{tder}_n)$ .

sder  $\leftrightarrow$  tree-level  $\mathcal{A}$   
tder  $\leftrightarrow$  more

$$F \in \mathcal{U}(\text{tder}_2); \quad F^{-1}e(x+y)F = e(x)e(y) \iff F \in \text{Solo}$$

$$\Phi = \Phi_F := (F^{12,3})^{-1}(F^{1,2})^{-1}F^{23}F^{1,23} \in \mathcal{U}(\text{sder}_3)$$

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4} = \Phi^{12,3,4}\Phi^{1,2,34} \quad \text{"the pentagon"}$$

$$t = \frac{1}{2}(y, x) \in \text{sder}_2 \text{ satisfies } 4T \quad \text{and} \quad r = (y, 0) \in \text{tder}_2 \text{ satisfies } 6T$$

$$R := e(r) \text{ satisfies Yang-Baxter: } R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

$$\text{also } R^{12,3} = R^{13}R^{23} \text{ and } F^{23}R^{1,23}(F^{23})^{-1} = R^{12}R^{13}$$

$$\tau(F) := RF^{21}e(-t) \text{ is an involution, } \Phi_{\tau(F)} = (\Phi_F^{321})^{-1}$$

$$\text{Sol}_0^r := \{F : \tau(F) = F\} \text{ is non-empty; for } F \in \text{Sol}_0^r,$$

$$e(t^{13} + t^{23}) = \Phi^{213}e(t^{13})(\Phi^{231})^{-1}e(t^{23})\Phi^{321}$$

$$\text{and } e(t^{12} + t^{13}) = (\Phi^{132})^{-1}e(t^{13})\Phi^{312}e(t^{12})\Phi$$



Alekseev

This is just a part of the Alekseev-Torossian work!

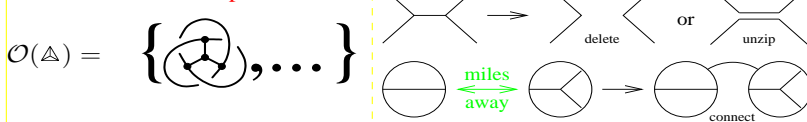


Torossian

- Related to the Kashiwara-Vergne Conjecture!
- Will likely lead to an explicit tree-level associator, a linear equation away from a 1-loop equation, two linear equations away from a 2-loop associator, etc.!
- A baby version of the QUEA equations; we may be on the right tracks!

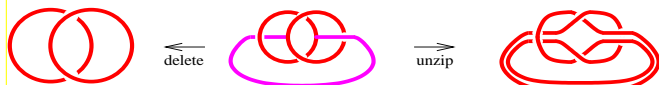
So What?

## Knotted Trivalent Graphs



**Theorem.** KTG is generated by the unknotted  $\Delta$  and the Möbius band, with identifiable relations between them.

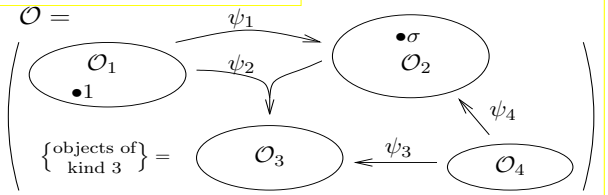
**Theorem.**  $Z(\Delta)$  is equivalent to an associator  $\Phi$ .



Algebraic Knot Theory

**Theorem.**  $\{\text{ribbon knots}\} \sim \{u\gamma : \gamma \in \mathcal{O}(\circ\circ), d\gamma = \circ\circ\}$ .

Hence an expansion for KTG may tell us about ribbon knots, knots of genus 5, boundary links, etc.



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Defining  $\text{proj } \mathcal{O}$ . The augmentation "ideal":

$$I = I_{\mathcal{O}} := \left\{ \begin{array}{l} \text{formal differences of ob-} \\ \text{jects "of the same kind"} \end{array} \right\}$$

Then  $I^n := \left\{ \begin{array}{l} \text{all outputs of algebraic} \\ \text{expressions at least } n \text{ of} \\ \text{whose inputs are in } I \end{array} \right\}$ , and

$$\text{proj } \mathcal{O} := \bigoplus_{n \geq 0} I^n / I^{n+1} \quad \left( \begin{array}{l} \text{has same kinds and opera-} \\ \text{tions, but different objects} \\ \text{and axioms} \end{array} \right)$$

## Knot Theory Anchors.

- $(\mathcal{O}/I^{n+1})^*$  is "type  $n$  invariants".
- $(I^n/I^{n+1})^*$  is "weight systems".
- $\text{proj } \mathcal{O}$  is  $\mathcal{A}$ , "chord diagrams".



Vassiliev  
Goussarov

## Warmup Examples.

- The projectivization of a group is a graded associative algebra.
- A quandle: a set  $Q$  with a binary op  $\wedge$  s.t.

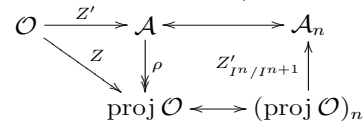
$$1 \wedge x = 1, \quad x \wedge 1 = x \wedge x = x, \quad (\text{appetizers})$$

$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \quad (\text{main})$$

$\text{proj } Q$  is a graded Lie algebra: set  $\bar{v} := (v - 1)$  (these generate  $I$ ), feed  $1 + \bar{x}, 1 + \bar{y}, 1 + \bar{z}$  in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

**An Expansion** is  $Z: \mathcal{O} \rightarrow \text{proj } \mathcal{O}$  s.t.  $Z(I^n) \subset (\text{proj } \mathcal{O})_{\geq n}$  and  $Z'_{I^n/I^{n+1}} = Id_{I^n/I^{n+1}}$  (A "universal finite type invariant"). In practice, it is hard to determine  $\text{proj } \mathcal{O}$ , but easy to guess a surjection  $\rho: \mathcal{A} \rightarrow \text{proj } \mathcal{O}$ . So find  $Z': \mathcal{O} \rightarrow \mathcal{A}$  with  $Z'(I^n) \subset \mathcal{A}_{\geq n}$  and  $Z'_{I^n/I^{n+1}} \circ \rho_n = Id_{\mathcal{A}_n}$ :



Can you make this diagram less confusing?



X-S. Lin

**Homomorphic Expansions** are expansions that intertwine the algebraic structure on  $\mathcal{O}$  and  $\text{proj } \mathcal{O}$ . They provide finite / combinatorial handles on global problems.

**The Key Point.** If  $\mathcal{O}$  is finitely presented, finding a homomorphic expansion is solving finitely many equations with finitely many unknowns, in some graded spaces.