

Projectivization, w-Knots, Kashiwara-Vergne and Alekseev-Torossian

"An Algebraic Structure"

The Categorification Speculative Paradigm.

- Every object in math is the Euler characteristic of a complex.
- Every operation in math lifts to an operation between complexes.
- Every identity in mathematics is true up to homotopy.

The Projectivization Tentative Speculative Paradigm. Projectivization?

- Every graded algebraic structure in mathematics is the projectivization of a plain ("global") one.
- Every equation written in a graded algebraic structure is an equation for a homomorphic expansion, or for an automorphism of such.

Graded Equations Examples

- $e(x+y) = e(x)e(y)$ in $\mathbb{Q}[[x, y]]$.
- The pentagon and hexagons in $\mathcal{A}(\uparrow_{3,4})$.
- The equations defining a QUEA, the work of Etingof and Kazhdan.

The Alekseev-Torossian equations in $\mathcal{U}(\text{sder}_n)$ and $\mathcal{U}(\text{tder}_n)$.

$$F \in \mathcal{U}(\text{tder}_2); \quad F^{-1}e(x+y)F = e(x)e(y) \iff F \in \text{Sol}_0$$

$$\Phi = \Phi_F := (F^{12,3})^{-1}(F^{1,2})^{-1}F^{23}F^{1,23} \in \mathcal{U}(\text{sder}_3)$$

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4} = \Phi^{12,3,4}\Phi^{1,2,34} \quad \text{"the pentagon"}$$

$$t = \frac{1}{2}(y, x) \in \text{sder}_2 \text{ satisfies } 4T \quad \text{and} \quad r = (y, 0) \in \text{tder}_2 \text{ satisfies } 6T$$

$$R := e(r) \text{ satisfies Yang-Baxter: } R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

$$\text{also } R^{12,3} = R^{13}R^{23} \text{ and } F^{23}R^{1,23}(F^{23})^{-1} = R^{12}R^{13}$$

$$\tau(F) := RF^{21}e(-t) \text{ is an involution, } \Phi_{\tau(F)} = (\Phi_F^{321})^{-1}$$

$$\text{Sol}_0^r := \{F: \tau(F) = F\} \text{ is non-empty; for } F \in \text{Sol}_0^r,$$

$$e(t^{13} + t^{23}) = \Phi^{213}e(t^{13})(\Phi^{231})^{-1}e(t^{23})\Phi^{321}$$

$$\text{and } e(t^{12} + t^{13}) = (\Phi^{132})^{-1}e(t^{13})\Phi^{312}e(t^{12})\Phi$$



Alekseev

This is just a part of the Alekseev-Torossian work!

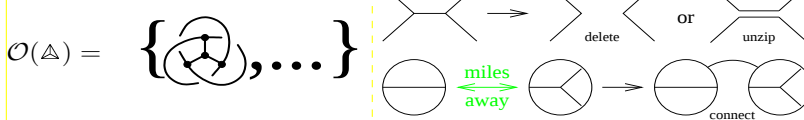


Torossian

- Related to the Kashiwara-Vergne Conjecture!
- Will likely lead to an explicit tree-level associator, a linear equation away from a 1-loop equation, two linear equations away from a 2-loop associator, etc.!
- A baby version of the QUEA equations; we may be on the right tracks!

So What?

Knotted Trivalent Graphs



Theorem. KTG is generated by the unknotted Δ and the Möbius band, with identifiable relations between them.

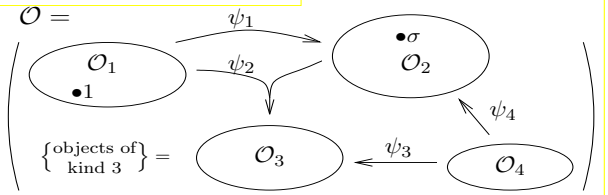
Theorem. $Z(\Delta)$ is equivalent to an associator Φ .



Algebraic Knot Theory

Theorem. $\{\text{ribbon knots}\} \sim \{u\gamma: \gamma \in \mathcal{O}(\circ\circ), d\gamma = \circ\circ\}$.

Hence an expansion for KTG may tell us about ribbon knots, knots of genus 5, boundary links, etc.



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Defining $\text{proj } \mathcal{O}$. The augmentation "ideal":

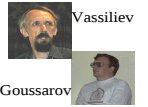
$$I = I_{\mathcal{O}} := \left\{ \begin{array}{l} \text{formal differences of ob-} \\ \text{jects "of the same kind"} \end{array} \right\}.$$

Then $I^n := \left\{ \begin{array}{l} \text{all outputs of algebraic} \\ \text{expressions at least } n \text{ of} \\ \text{whose inputs are in } I \end{array} \right\}$, and

$$\text{proj } \mathcal{O} := \bigoplus_{n \geq 0} I^n / I^{n+1} \quad \left(\begin{array}{l} \text{has same kinds and opera-} \\ \text{tions, but different objects} \\ \text{and axioms} \end{array} \right).$$

Knot Theory Anchors.

- $(\mathcal{O}/I^{n+1})^*$ is "type n invariants".
- $(I^n/I^{n+1})^*$ is "weight systems".
- $\text{proj } \mathcal{O}$ is $\mathcal{A}$, "chord diagrams".



Vassiliev
Goussarov

Warmup Examples.

- The projectivization of a group is a graded associative algebra.
- A quandle: a set Q with a binary op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x \wedge x = x, \quad (x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z).$$

$\text{proj } Q$ is a graded Lie algebra: set $\bar{v} := (v-1)$ (these generate I), feed $1+\bar{x}, 1+\bar{y}, 1+\bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

An Expansion is $Z: \mathcal{O} \rightarrow \text{proj } \mathcal{O}$ s.t. $Z(I^n) \subset (\text{proj } \mathcal{O})_{\geq n}$ and $Z'_{I^n/I^{n+1}} = Id_{I^n/I^{n+1}}$ (A "universal finite type invariant"). In practice, it is hard to determine $\text{proj } \mathcal{O}$, but easy to guess a surjection $\rho: \mathcal{A} \rightarrow \text{proj } \mathcal{O}$. So find $Z': \mathcal{O} \rightarrow \mathcal{A}$ with $Z'(I^n) \subset \mathcal{A}_{\geq n}$ and $Z'_{I^n/I^{n+1}} \circ \rho_n = Id_{\mathcal{A}_n}$:

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{Z'} & \mathcal{A} \\ & \searrow Z & \downarrow \rho \\ & & \text{proj } \mathcal{O} \end{array} \quad \begin{array}{c} \xleftarrow{\quad} \mathcal{A}_n \\ \uparrow Z'_{I^n/I^{n+1}} \\ \text{proj } \mathcal{O} \end{array} \xleftarrow{\quad} (\text{proj } \mathcal{O})_n$$

Can you make this diagram less confusing?



X-S. Lin

Homomorphic Expansions are expansions that intertwine the algebraic structure on $\mathcal{O}$ and $\text{proj } \mathcal{O}$. They provide finite / combinatorial handles on global problems.

The Key Point. If $\mathcal{O}$ is finitely presented, finding a homomorphic expansion is solving finitely many equations with finitely many unknowns, in some graded spaces.