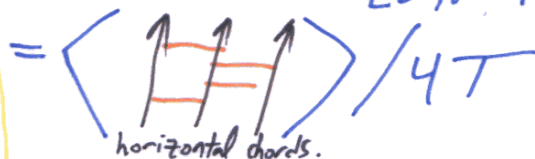


Math 1352 Algebraic Knot Theory - The Knizhnik-Zamolodchikov Connection

Theorem 1. The following is an invariant of braids in $\mathbb{R}^2 \times \mathbb{C}^n$ (Fixed endpoints)

$$Z(B) = \oint \frac{Dp}{(2\pi i)^m} \bigwedge_{i=1}^m \frac{dz_i - dz_i'}{z_i - z_i'} \quad \text{in } A(1_n) := \langle t^{ij} : 1 \leq i \neq j \leq n \rangle / \begin{matrix} t^{ij} = t^{ji} \\ [t^{ij}, t^{kl}] = 0 \\ [t^{ij}, t^{ik} + t^{jk}] = 0 \end{matrix}$$

$t_1 \leq \dots \leq t_m$
 $p = (z_i, z_i')$



"Chord diagrams for braids".

Formal Connection
& Curvature.

Let $\Omega \in \Omega^1(M, A)$ with $\deg \Omega = 1$.

$\gamma: [0, 1] \rightarrow M$ induces

$\phi: \Delta^m = \{0 \leq t_1 \leq \dots \leq t_m \leq 1\} \rightarrow M^m$.

Set $\text{hol}_\gamma(\Omega) = \text{Pexp} \int_\gamma \Omega = \oint_{\Delta^m} \phi^* \Omega^m$

where $\Omega^m := \pi_1^* \Omega \wedge \dots \wedge \pi_m^* \Omega$

Theorem 2. If $F_\Omega := d\Omega + \Omega \wedge \Omega = 0$, then $\text{hol}_\gamma(\Omega)$ is invariant under end-point preserving homotopies of γ .

The KZ connection.

$M = \mathbb{C}^n \setminus \{\text{diagonals}\}$, $A = A(1_n)$,

and $\Omega = \sum_{i < j} t^{ij} w_{ij}$ where $w_{ij} = \frac{dz_i - dz_j}{z_i - z_j} = d \log(z_i - z_j)$ locally

Compute $F_\Omega = d\Omega + \Omega \wedge \Omega$: $dw_{ij} = 0$ so $d\Omega = 0$.

$\Omega \wedge \Omega = \sum_{i < j, k < l} t^{ij} t^{kl} w_{ij} \wedge w_{kl} = A + B + C$ where

$A = C = 0$ as $[t^{ij}, t^{kl}] = 0$ if $|i, j, k, l| = 2$ or 4 and

$B = \sum_{\alpha < \beta} [t^{\alpha\beta}, t^{\beta\alpha}] w_{\alpha\beta} \wedge w_{\beta\alpha} + \text{cyclic perms}$

$= \sum_{\alpha < \beta} \gamma^{\alpha\beta} (w_{\alpha\beta} \wedge w_{\beta\alpha} + \text{cyclic perms}) = 0$ by Arnold's identity

Note: by 4T, $[t^{\alpha\beta}, t^{\beta\alpha}] = \gamma^{\alpha\beta} = \text{diagram}$

Proof of 1

Simply take in theorem 2, $\gamma =$ the braid and

$\Omega =$ the KZ Connection.

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