FUNDAMENTAL CONCEPTS IN DIFFERENTIAL GEOMETRY
FALL 2000
EXERCISES HANDOUT # 8

1. Exercises for the Proper Course

1. Let
$$H^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}.$$
Show that every diffeomorphism $f\colon H \to H$ maps the boundaries diffeomorphically.

Deduce that the boundary of a manifold is indeed well defined and is a manifold in its own right.

2. Let M be a smooth manifold (without boundary) and let $g\colon M \to \mathbb{R}$ be a smooth function which has 0 as a regular value. Show that
$$X = g^{-1}((-\infty, 0])$$
is a smooth manifold whose boundary is $g^{-1}(0)$.

Conclude that the unit disk D^m is a smooth manifold whose boundary is S^{m-1}.

Hint: the proof is nearly identical to the proof that $g^{-1}(0)$ is a smooth manifold given in class long ago.

3. Let $f\colon X^m \to N^n$ be a smooth map where X has a boundary and N does not. Suppose that $y \in N$ is a regular value, both for f and for $f|_{\partial X}$. Show that $f^{-1}(y)$ is a smooth $(m - n$-manifold with boundary. Furthermore the boundary $\partial(f^{-1}(0))$ is precisely equal to the intersection $f^{-1}(y) \cap \partial X$.

4. Let M be a compact manifold with boundary. Prove that there is no smooth map $f\colon M \to \partial M$ that leaves ∂M pointwise fixed.

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