Knot Theory: Notes for Lecture 4 (24.4.2001)

Based on lectures given by Prof. Dror Bar-Natan
Taken by Shlomi Kotler

May 26, 2001

The Bi-Algebra Structure of $\mathcal{A}$

Our last discussion ended with the following statements:

1. Let V_i be a finite type knot invariant of type n_i, for $i \in \{1, 2\}$, then the invariant $V_1 \cdot V_2$ is of type $n_1 + n_2$.

2. There exist a unique operator $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that the following holds:

$$W_{V_1 \cdot V_2} = (W_{V_1} \otimes W_{V_2}) \circ \Delta$$

   We remind that the notation W_V means the weight system corresponding to V, thus:

$$W_{V_1 \cdot V_2} : \mathcal{A} \to \mathbb{F}$$

   and

$$W_{V_1} \otimes W_{V_2} : \mathcal{A} \otimes \mathcal{A} \to \mathbb{F}$$

   are linear functionals.

We start by considering the invariant $V_1 \cdot V_2$ as it is defined on non-singular knots by multiplication of the values of V_1 and V_2 calculated on the specific knot. To calculate $V_1 \cdot V_2$ on a singular knot we need to express the calculation as a calculation of $V_1 \cdot V_2$ on non-singular knots and then calculate each term by the simple multiplication. As an example, let us calculate:

$$\begin{aligned}
(V_1 \cdot V_2)(\doublepoint\doublepoint) &= (V_1 \cdot V_2)(\overcrossing\doublepoint) - (V_1 \cdot V_2)(\undercrossing\doublepoint) \\
&= (V_1 \cdot V_2)(\overcrossing\overcrossing) - (V_1 \cdot V_2)(\overcrossing\undercrossing) - (V_1 \cdot V_2)(\undercrossing\overcrossing) + (V_1 \cdot V_2)(\undercrossing\undercrossing) \\
&= V_1(\overcrossing\overcrossing) \cdot V_2(\overcrossing\overcrossing) - V_1(\overcrossing\undercrossing) \cdot V_2(\overcrossing\undercrossing) - V_1(\undercrossing\overcrossing) \cdot V_2(\undercrossing\overcrossing) + V_1(\undercrossing\undercrossing) \cdot V_2(\undercrossing\undercrossing)
\end{aligned}$$

By adding and subtracting the term $V_1(\times\times) \cdot V_2(\times\times)$ and noticing that $V_1(\times\times) \cdot V_2(\times\times) - V_1(\times\times) \cdot V_2(\times\times)$ equals $V_1(\times\times) \cdot V_2(\times\times)$ we get:

$$
\begin{aligned}
= \quad & (V_1(\times\times) \cdot V_2(\times\times) - V_1(\times\times) \cdot V_2(\times\times)) + V_1(\times\times) \cdot V_2(\times\times) \\
& +V_1(\times\times) \cdot V_2(\times\times) - V_1(\times\times) \cdot V_2(\times\times) + V_1(\times\times) \cdot V_2(\times\times) \\
= \quad & V_1(\times\times) \cdot V_2(\times\times) + V_1(\times\times) \cdot V_2(\times\times) + \\
& V_1(\times\times) \cdot V_2(\times\times) - V_1(\times\times) \cdot V_2(\times\times) + V_1(\times\times) \cdot V_2(\times\times) =
\end{aligned}
$$

By using this simple algebra we will arrive at the expression:

$$V_1(\times\times) \cdot V_2(\times\times) + V_1(\times\times) \cdot V_2(\times\times) + V_1(\times\times) \cdot V_2(\times\times) + V_1(\times\times) \cdot V_2(\times\times)$$

For the general case of n singular points we spend our next few breaths to sharpen our algebraic machinery.

Consider a fixed knot K with n singular points. Suppose we number these points arbitrarily. An element in $\{|+\rangle, |-\rangle\}^n$ describes a single regulation of the singular knot: each entry in such an element tells us to break the singular point to an over-crossing (if the entry is $|+\rangle$) or an under-crossing (if it is $|-\rangle$). The entire element tells us how to transform the singular knot into a regular knot (which is the source of the name of the process: regulation). Given V we may define a linear functional f_V on the linear space $L^{\otimes n} = \langle\{|+\rangle, |-\rangle\}^n\rangle$ by defining its values on the basis as the values of V on the corresponding regular knots.

A singular knot which was created by regulating some of the points in K can be represented as an element of $L^{\otimes n}$. For examples, take $n = 1$ and represent x as $D = |+\rangle - |-\rangle$ or take $n = 2$ and represent ox as $o \otimes o - o \otimes u$ which is $o \otimes D$. The calculation of V on such a singular knot is simple: it is the value of f_V on the corresponding element in $L^{\otimes n}$.

Generally speaking, given two functions on $f, g : L^{\otimes n} \to F$ we define the multiplication $f \cdot g : L^{\otimes n} \to F$ by defining it on the basis of the vector space as the multiplication result of f and g calculated on the basis, and linearly extending $f \cdot g$ to the whole space. But if we define an auxiliary function $\Delta_n : L^{\otimes n} \to L^{\otimes n} \otimes L^{\otimes n}$ by defining it on the basis of $L^{\otimes n}$ as $\Delta_n(l) = l \otimes l$, we may express:

$$(f \cdot g)(v) = (f \otimes g)(\Delta_n(v))$$

The spaces $L^{\otimes n} \otimes L^{\otimes n}$ and $(L \otimes L)^{\otimes n}$ are naturally isomorphic in a way that the following:

$$
\begin{array}{ccc}
L^{\otimes n} & \xrightarrow{\Delta_1^{\otimes n}} & (L \otimes L)^{\otimes n} \\
\| & & \cong\downarrow \\
L^{\otimes n} & \xrightarrow{\Delta_n} & L^{\otimes n} \otimes L^{\otimes n}
\end{array}
$$

commutes. From this we deduce that:

$$\Delta_n(v_1 \otimes \ldots \otimes v_n) = \Delta_1(v_1) \otimes \ldots \otimes \Delta_1(v_n)$$

We now use these tools to calculate $V_1 \cdot V_2$ on the singular knot of n singular points. This in turn equals the value of $f_{V_1 \cdot V_2}$ on $\underbrace{D \otimes \ldots \otimes D}_{n}$:

$$\begin{array}{ll} f_{V_1 \cdot V_2}(D \otimes \ldots \otimes D) & = \\ (f_{V_1} \cdot f_{V_2})(D \otimes \ldots \otimes D) & = \\ f_{V_1} \otimes f_{V_2}(\Delta_n(D \otimes \ldots \otimes D)) & = \\ f_{V_1} \otimes f_{V_2}(\Delta_1(D) \otimes \ldots \otimes \Delta_1(D)) & \end{array}$$

but we may write $\Delta_1(D)$ as:

$$\Delta_1(D) = |++\rangle - |--\rangle = |++\rangle - |+-\rangle + |+-\rangle - |--\rangle = |+\rangle \otimes D + D \otimes |-\rangle$$

so we have:

$$\begin{array}{ll} f_{V_1 \cdot V_2}(D \otimes \ldots \otimes D) & = \\ (f_{V_1} \otimes f_{V_2})((|+\rangle \otimes D + D \otimes |-\rangle)^{\otimes n}) & = \\ \sum_{I \subset \{1 \ldots n\}} f_{V_1}(v_I) \cdot f_{V_2}(w_I) & \end{array}$$

where v_I is the element of $L^{\otimes n}$ composed of $|+\rangle$'s in the places whose index is contained in I tensored with D 's in complement places, and w_I is composed of D 's in the I places and $|-\rangle$ in the I^c places.

We may now write the equivalent formula for knot invariants:

$$(V_1 \cdot V_2)(\underbrace{x \ldots x}_{n}) = \sum_{I \subset \{1 \ldots n\}} V_1(K_I^1) \cdot V_2(K_I^2)$$

where K_I^1 (corresponds to v_I) is the knot which results from breaking the singular points of K whose index is in I to over-crossings and K_I^2 (corresponds to w_I) is the knot we get from breaking the singular points of K whose index is in I^c to under-crossings.

Returning to the conditions of our question, if we feed $V_1 \cdot V_2$ with an $n_1 + n_2$ singular points knot, most elements in the above sum vanish and in fact we sum only on those I which satisfy $|I| = n_2$. It is clear that this sum vanishes totally if fed with more than $n_1 + n_2$ singular points.

To write this in the language of weight systems on chord diagrams, we define $D_{K_I^1}$ as the chord diagram of K (denoted as D_K) having the chords corresponding to I removed and $D_{K_I^2}$ as the one having the chords corresponding to I^c removed. We may now equate:

$$W_{V_1 \cdot V_2}(D_K) = \sum_{|I| = n_2} W_{V_1}(D_{K_I^1}) \cdot W_{V_2}(D_{K_I^2})$$

The last equation directs us on how to define $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$:

$$\Delta(C) \stackrel{def}{=} \sum_{I \subset \{\text{chords of C}\}} C_I \otimes C_{I^c}$$

Of course, this definition is defined on chord diagrams. We need to show that it is well defined on chord diagrams modulo $4-T$.

Lemma 0.1. *$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is well defined and is co-commutative and co-associative.*

Proof. We start by considering the $4T$ relation, on which we expect Δ to equal zero (fig 1).

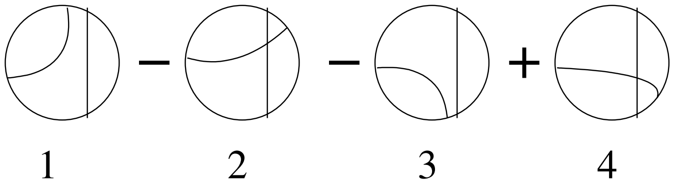

Figure 1: 4T relation

The result of activating Δ on diagram 1 in fig. 1 is the four diagrams in fig. 2

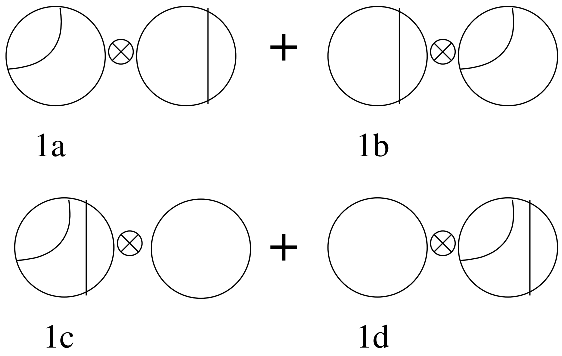

Figure 2: the first four terms of $\Delta(4T)$

The mixed terms of $\Delta(1)$ ($1a$ and $1b$) will cancel out with the mixed terms of $\Delta(2)$. The remaining $1c$ and $1d$ will cancel out with those of $\Delta(3)$. Counting the contribution of the rest of the terms will lead to a sum of zero.

The co-commutativity of Δ results straight from its formula, since summing over I is equivalent to summing over I^c. The definition of Δ also enforces co-associativity, from similar considerations. $\square$

Definition 0.1. *A 5-tuple $(A, m, \epsilon, \Delta, \eta)$ is called a bi-algebra (or a Hopf Algebra) if (A, m, ϵ) is an algebra and (A, Δ, η) is a co-algebra and the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\Delta"] & \mathcal{A} \otimes \mathcal{A} \\
\mathcal{A} \otimes \mathcal{A} \arrow[u, "m"] \arrow[r, "\sigma_{2,3} \circ \Delta \otimes \Delta"] & (\mathcal{A} \otimes \mathcal{A}) \otimes (\mathcal{A} \otimes \mathcal{A}) \arrow[u, "m \otimes m"']
\end{tikzcd}$$

where $\sigma_{2,3} : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ takes the base elements $(v_1 \otimes v_2 \otimes v_3 \otimes v_4)$ to $(v_1 \otimes v_3 \otimes v_2 \otimes v_4)$.

Lemma 0.2. *$\mathcal{A}$ is a bi-algebra.*

Proof. This is proved by the diagram in fig. 3 on the base elements of $\mathcal{A} \otimes \mathcal{A}$.

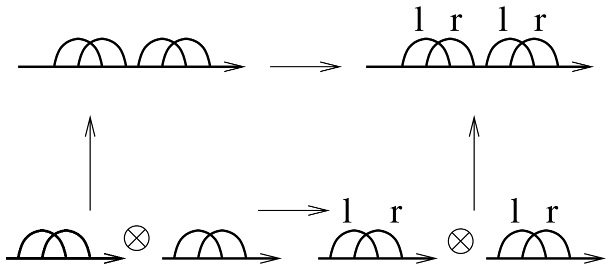

Figure 3: proof that $\mathcal{A}$ is a bi-algebra

Beginning with the lower-left sketch, we have an element $v \otimes w$ of $\mathcal{A} \otimes \mathcal{A}$. Going up, we glue it together using m to a single diagram and turning to the upper-right corner we activate Δ resulting with a summation over all possible splittings of the unified diagram into tensor products of two sub diagrams, each the complement of the other. This is sketched schematically by the l and r letters, telling us to take two portions of the diagram (marked as l) - one of which originated from v and the other from w- and unifying them into a diagram and tensoring it with the unification of the other two portions (marked as r).

Now doing the calculation in the opposite direction we first split the elements v and w individually and then unify the splittings according to the letters l and r as before. We clearly get the same result. □

$\mathcal{A}$ and Lie Algebras

One of the most important issues in knot theory is its relation to the theory of Lie Algebras. The deepness and richness of an area in mathematics usually follows with strong connections to other areas, and it will be exciting to follow through an example of such a case.

Definition 0.2. *A Lie Algebra L is a vector space along with a bi-linear map $[,] : L \otimes L \to L$ such that:*

1. *$[,]$ is anti-symmetric, i.e. $[x,y] = -[y,x]$ for all $x, y \in L$.*
2. *The Jacobi relation holds:*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

for all $x, y, z \in L$.

As an example we can take L to be $M_n(\mathbb{F})$ or, in fact, any associative algebra A, with the commutator as the bracket operator: $[x,y] \stackrel{def}{=} xy - yx$.

Expressions in a Lie algebra can be expressed using diagrams instead of formulas. This can be understood without a formal definition from fig. 4.

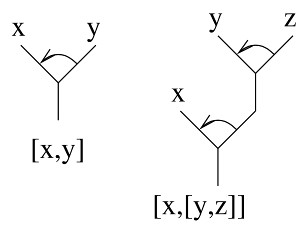

Figure 4: examples of expressing terms in a lie algebra with diagrams

We express the anti-symmetry property of the Lie bracket by defining the sum of two diagrams which differ only by a single vertex direction as zero (fig. 5.

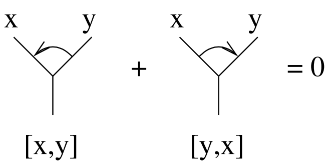

Figure 5: example 1 of the anti-symmetry relation in diagrams

A slightly more complex example is in fig. 6.

We enforce the Jacobi relation on our diagrams by the IHX relation, as seen in fig. 7. Following through the diagram renders the Jacobi relation as requested.

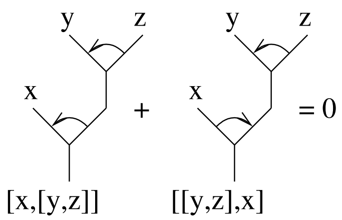

Figure 6: example 2 of the anti-symmetry relation in diagrams

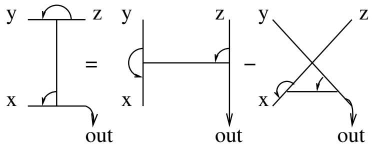

Figure 7: the Jacobi relation in diagrams (the IHX relation)

Returning to our example of an associative algebra as a Lie algebra with the commutator as the Lie bracket we may denote expressions in the algebra as in the diagram in fig. 8.

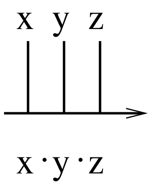

Figure 8: associative algebra in diagrams

We enforce a relation on diagrams of Lie algebras so as to translate them into diagrams of associative algebras. This relation is called the STU relation, shown in fig. 9.

Using a pencil and an eraser one can convince himself that the STU relation implies the IHX relation. In other words, the translation rule STU gives a structure of Lie algebra diagrams given a structure of diagrams of an associative algebra.

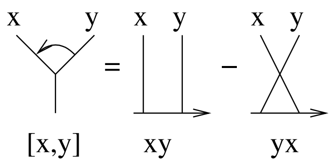

Figure 9: STU relation