11: Find all numbers x for which

(ii) $|x-3|<8$

solution:

$$|x-3|<8 \Leftrightarrow -8<x-3<8 \Leftrightarrow -8+3<x<8+3 \Leftrightarrow -5<x<11$$

(iv) $|x-1|+|x-2|>1$

solution:

$$|x-1|+|x-2| = 1-x+2-x = 3-2x \quad x\in(-\infty,1)$$
$$|x-1|+|x-2| = x-1+2-x = 1 \quad x\in[1,2]$$
$$|x-1|+|x-2| = x-1+x-2 = 2x-3 \quad x\in(2,\infty)$$
$$\therefore$$
$$|x-1|+|x-2|>1 \Leftrightarrow 3-2x>1 \Leftrightarrow -2x>-2 \Leftrightarrow x<1 \quad x\in(-\infty,1)$$
$$|x-1|+|x-2|>1 \Leftrightarrow 1<1 \Leftrightarrow x\in\varnothing \quad x\in[1,2]$$
$$|x-1|+|x-2|>1 \Leftrightarrow 2x-3>1 \Leftrightarrow x>2 \quad x\in(2,\infty)$$
$$\textit{thus}$$
$$x\in(-\infty,1)\cup(2,\infty)$$

(vi) $|x-1|+|x+1|<1$

solution:

$$\textit{if} \;\; x\in(-\infty,-1) \quad |x-1|+|x+1|<1 \Leftrightarrow 1-x+(-x-1)<1 \Leftrightarrow x>\frac{1}{2} \Leftrightarrow x\in\varnothing$$
$$\textit{if} \;\; x\in[-1,1] \quad |x-1|+|x+1|<1 \Leftrightarrow 1-x+(x+1)<1 \Leftrightarrow 2<1 \Leftrightarrow x\in\varnothing$$
$$\textit{if} \;\; x\in(-\infty,-1) \quad |x-1|+|x+1|<1 \Leftrightarrow (x-1)+(x+1)<1 \Leftrightarrow x<\frac{1}{2} \Leftrightarrow x\in\varnothing$$
$$\textit{thus}$$
$$x\in\varnothing$$

(viii) $|x-1|\cdot|x+2|=3$

solution:

$$\because \quad |a|\cdot|b| = |a\cdot b|$$
$$\therefore \quad |x-1|\cdot|x+2| = |(x-1)\cdot(x+2)| = |x^2+x-2|$$
$$\because \quad |x-1|\cdot|x+2| = 3$$
$$\therefore \quad |x^2+x-2| = 3 \Leftrightarrow x^2+x-2=\pm 3 \Leftrightarrow x^2+x-2\mp 3=0$$
$$\textit{thus} \quad x^2+x+1=0 \;\textit{or}\; x^2+x-5=0$$
$$x^2+x+1=0 \Leftrightarrow x\in\varnothing \quad x^2+x-5=0 \Leftrightarrow x=\frac{-1\pm\sqrt{21}}{2}$$
$$\therefore \quad x=\frac{-1\pm\sqrt{21}}{2}$$

12 Prove the following

(ii) $\left|\frac{1}{x}\right| = \frac{1}{|x|} \quad if\ x \neq 0$

prove:

$as\ we\ proved\ before, \quad |ab| = |a||b|$

$then\ let \quad a = x, b = \frac{1}{x} \quad \Leftrightarrow \quad \left|x \cdot \frac{1}{x}\right| = |x| \cdot \left|\frac{1}{x}\right|$

$\because \quad x \neq 0$

$\therefore \quad divide\ equation\ with\ |x|,\ we\ got \quad \frac{\left|x \cdot \frac{1}{x}\right|}{|x|} = \left|\frac{1}{x}\right|$

$\because \quad \left|x \cdot \frac{1}{x}\right| = 1$

$\therefore \quad \frac{1}{|x|} = \left|\frac{1}{x}\right| \quad QED$

(iv) $|x - y| \leq |x| + |y|$

prove:

$as\ we\ proved\ before, \quad |a + b| \leq |a| + |b|$

$\therefore \quad let\ a = x, b = -y\ we\ got \quad |x + (-y)| \leq |x| + |-y|$

$thus \quad |x - y| \leq |x| + |y| \quad QED$

(vi) $|(|x| - |y|)| \leq |x - y|$

prove:

$as\ we\ proved, |a + b| \leq |a| + |b|$

$let\ a = x - y, b = y \quad we\ have\ |(x - y) + y| \leq |x - y| + |y|$

$thus\ |x| \leq |x - y| + |y| \Leftrightarrow |x| - |y| \leq |x - y|$

$if\ |x| < |y|, |(|x| - |y|)| = |y| - |x| \leq |y - x| = |x - y|$

$if\ |x| \geq |y|, |(|x| - |y|)| = |x| - |y| \leq |x - y|$

$\therefore \quad |(|x| - |y|)| \leq |x - y| \quad QED$

14 (a) Prove that |a| = |-a|.

prove:

$if\ a \in (-\infty, 0], \quad |a| = -a\ and\ |-a| = -a \Leftrightarrow |a| = |-a|$

$if\ a \in [0, \infty), \quad then\ (-a) \in (-\infty, 0],\ from\ above\ we\ have\ |-a| = |-(-a)| = |a|$

$\therefore for\ a \in (-\infty, \infty), |a| = |-a| \quad QED$

(b) Prove that $-b \leq a \leq b$ if and only if $|a| \leq b$.

Prove:

$as\ we\ proved, \quad -|a| \leq a \leq |a|$

$\therefore |a| \leq b \Leftrightarrow -|a| \geq -b \Leftrightarrow -b \leq -|a| \leq a \leq |a| \leq b$

$thus \quad |a| \leq b \Leftrightarrow -b \leq a \leq b \quad QED$

(c) Use this fact to give a new proof that $|a+b| \le |a| + |b|$

prove:

$$\because -|a| \le a \le |a|, -|b| \le b \le |b|$$
$$\therefore -|a| + (-|b|) \le a+b \le |a| + |b|$$
$$thus \quad -(|a| + |b|) \le a+b \le |a| + |b|$$
$$from\ the\ proof\ of\ (b), we\ know$$
$$|a+b| \le |a| + |b|$$
$$QED$$

1. Show that if a > 0, then $ax^2 + bx + c \ge 0$ for all value of x if and only if $b^2 - 4ac \le 0$

Prove:

$$\because a > 0$$
$$\therefore ax^2 + bx + c \ge 0 \Leftrightarrow x^2 + \frac{b}{a}x + \frac{c}{a} \ge 0 \Leftrightarrow x^2 + 2 \cdot \frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} \ge 0$$
$$\Leftrightarrow \left(x + \left(\frac{b}{2a}\right)\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} \ge 0 \Leftrightarrow \left(x + \left(\frac{b}{2a}\right)\right)^2 - \frac{b^2 - 4ac}{4a^2} \ge 0$$
$$\because \left(x + \left(\frac{b}{2a}\right)\right)^2 \ge 0$$
$$\therefore \left(x + \left(\frac{b}{2a}\right)\right)^2 - \frac{b^2 - 4ac}{4a^2} \ge 0 \Leftrightarrow -\frac{b^2 - 4ac}{4a^2} \ge 0$$
$$\because a > 0 \Rightarrow -4a^2 < 0$$
$$\therefore -\frac{b^2 - 4ac}{4a^2} \ge 0 \Leftrightarrow b^2 - 4ac \le 0 \cdot \left(-4a^2\right) = 0$$
$$QED$$

2. Prove the Cauchy-Schwartz inequality

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le \left(a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2\right) \cdot \left(b_1{}^2 + b_2{}^2 + \cdots + b_n{}^2\right)$$

in two different ways:

(a) use $2xy \le x^2 + y^2$, with

$$x = \frac{|a_i|}{\sqrt{a_1{}^2 + a_2{}^2 + \cdots + a_n{}^2}} \quad y = \frac{|b_i|}{\sqrt{b_1{}^2 + b_2{}^2 + \cdots + b_n{}^2}}$$

Prove:

$\because (x-y)^2 \ge 0 \Leftrightarrow x^2 - 2xy + y^2 \ge 0 \Leftrightarrow x^2 + y^2 \ge 2xy$

$$let\ x = \frac{|a_i|}{\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}} \quad y = \frac{|b_i|}{\sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}}$$

$$then\ \frac{a_i^2}{a_1^2 + a_2^2 + \cdots + a_n^2} + \frac{b_i^2}{b_1^2 + b_2^2 + \cdots + b_n^2} \ge 2\frac{|a_i b_i|}{\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \cdot \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}}$$

$$\therefore \frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n a_i^2} + \frac{\sum_{i=1}^n b_i^2}{\sum_{i=1}^n b_i^2} \ge 2\frac{\sum_{i=1}^n |a_i b_i|}{\sqrt{\sum_{i=1}^n a_i^2} \cdot \sqrt{\sum_{i=1}^n b_i^2}}$$

$$thus\ 2 \ge 2\frac{\sum_{i=1}^n |a_i b_i|}{\sqrt{\sum_{i=1}^n a_i^2} \cdot \sqrt{\sum_{i=1}^n b_i^2}}$$

$$\because \sqrt{\sum_{i=1}^n a_i^2} \cdot \sqrt{\sum_{i=1}^n b_i^2} > 0$$

$$\therefore 2\sqrt{\sum_{i=1}^n a_i^2} \cdot \sqrt{\sum_{i=1}^n b_i^2} \ge 2\sum_{i=1}^n |a_i b_i| \Leftrightarrow \sqrt{\sum_{i=1}^n a_i^2} \cdot \sqrt{\sum_{i=1}^n b_i^2} \ge \sum_{i=1}^n |a_i b_i| \Leftrightarrow \sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2 \ge \left(\sum_{i=1}^n |a_i b_i|\right)^2$$

$$\because |a_i b_i| \ge a_i b_i$$

$$\therefore \sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n b_i^2 \ge \left(\sum_{i=1}^n |a_i b_i|\right)^2 \ge \left(\sum_{i=1}^n a_i b_i\right)^2$$

QED

(b) Consider the expression

$$(a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \cdots + (a_n x + b_n)^2$$

collect terms, and apply the result of Problem 1.

Prove:

$$(a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \cdots + (a_n x + b_n)^2 \ge 0$$
$$\Leftrightarrow \left(a_1^2 x^2 + 2a_1 b_1 x + b_1^2\right) + \left(a_2^2 x^2 + 2a_2 b_2 x + b_2^2\right) + \cdots + \left(a_n^2 x^2 + 2a_n b_n x + b_n^2\right) \ge 0$$
$$\Leftrightarrow \left(a_1^2 + a_2^2 + \cdots a_n^2\right)x^2 + 2(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)x + \left(b_1^2 + b_2^2 + \cdots b_n^2\right) \ge 0$$
$$\because \left(a_1^2 + a_2^2 + \cdots a_n^2\right) \ge 0$$
$$\therefore if \quad \left(a_1^2 + a_2^2 + \cdots a_n^2\right) > 0, \text{ apply the result of Problem1, we know}$$
$$\left(2(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)\right)^2 - 4\left(a_1^2 + a_2^2 + \cdots a_n^2\right)\left(b_1^2 + b_2^2 + \cdots b_n^2\right) \le 0$$
$$\Leftrightarrow 4(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \le 4\left(a_1^2 + a_2^2 + \cdots a_n^2\right)\left(b_1^2 + b_2^2 + \cdots b_n^2\right)$$
$$\Leftrightarrow (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \le \left(a_1^2 + a_2^2 + \cdots a_n^2\right)\left(b_1^2 + b_2^2 + \cdots b_n^2\right)$$
$$if \quad \left(a_1^2 + a_2^2 + \cdots a_n^2\right) = 0, \quad then\ a_1 = a_2 = \cdots = a_n = 0$$
$$\Leftrightarrow (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 = \left(a_1^2 + a_2^2 + \cdots + a_n^2\right) \cdot \left(b_1^2 + b_2^2 + \cdots + b_n^2\right) = 0$$

QED