University of Toronto

Department of Mathematics

Math 157 Exam 4

Monday, March 25, 2002

One hour and fifty minutes

There are five problems, each worth 20 points although they do not have equal difficulty. Write your answer in the space below the problem; use the back of the sheets and the last page for scratch paper. Write your name on each page.

No calculators

1	/20
2	/20
3	/20
4	/20
5	/20
Total	/100

1. Evaluate the following indefinite integrals (a "$+\,C$" will be assumed in your answer, so you do not need to write it):

   (a) $\displaystyle\int \frac{x\exp(\arctan(x))}{(1+x^2)^{3/2}}\,dx$

   (b) $\displaystyle\int \frac{2x+3}{(6x+7)^3}\,dx$

   (c) $\displaystyle\int \frac{dx}{x(x^2+1)}$

   (d) $\displaystyle\int x\,(\log x)^2\,dx$

   (e) $\displaystyle\int \arcsin x\,dx$

2. Do the following series converge (does the sum exist)? Explain briefly why or why not.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^2+1}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^2}$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+\log n}$

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\,\log n}$

3. Let $\{a_n\}_{n=1}^{\infty}$ be a positive and nonincreasing sequence.

   (a) Prove that if $\sum a_n$ exists, then $\lim n\, a_n = 0$.

   (b) Prove that the converse is false by giving a counterexample.

4. Let $f_0(x)$ be a continuous function on $[0, a]$ $(0 \leq x \leq a)$, and define

$$f_n(x) \;=\; \int_0^x f_{n-1}(t)\,dt, \qquad n = 1, 2, \ldots$$

Prove that

$$\lim_{n \to \infty} f_n(x) \;=\; 0 \qquad \text{for each } x \in [0, a].$$

5. An angel-food cake pan has the form of a circular trough with a parabolic cross-section. Let the radius of the centerline be R, the height be H, and the width W (outer radius minus inner radius). If the cross-section is represented as $y = f(x)$, where x is the distance from the rotation axis, then we may write (putting the curved side up)

$$f(x) \;=\; H \;-\; \frac{4H}{W^2}\left(x - R\right)^2, \qquad \text{for } R - \frac{W}{2} \le x \le R + \frac{W}{2}.$$

   (a) Draw a picture, and show that this is the right formula by evaluating $f(R-(W/2))$, $f(R)$, and $f(R+(W/2))$.

   (b) Calculate the area of the cross-section, and the volume of the cake.

   (c) Write the surface area as an integral; you do not need to evaluate the integral. Since you would use this for the purpose of deciding how much frosting to buy, do not include the flat bottom of the cake.