Dror Bar-Natan: Classes: 2004-05: Math 157 - Analysis I:

The 13 Postulates

Everything you ever wanted to know about the real numbers is summarized as follows. There is a set $\mathbb{R}$ "of real numbers" with two binary operations defined on it, $+$ and $\cdot$ ("addition" and "multiplication"), two different distinct elements 0 and 1 and a subset $\mathbb{P}$ "of positive numbers" so that the following 13 postulates hold:

P1 Addition is associative: $\forall a, b, c \quad a + (b + c) = (a + b) + c$ ("$\forall$" means "for every").

P2 The number 0 is an additive identity: $\forall a \quad a + 0 = 0 + a = a$.

P3 Additive inverses exist: $\forall a\ \exists(-a)$ s.t. $a + (-a) = (-a) + a = 0$ ("$\exists$" means "there is" or "there exists").

P4 Addition is commutative: $\forall a, b \quad a + b = b + a$.

P5 Multiplication is associative: $\forall a, b, c \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

P6 The number 1 is a multiplicative identity: $\forall a \quad a \cdot 1 = 1 \cdot a = a$.

P7 Multiplicative inverses exist: $\forall a \neq 0\ \exists a^{-1}$ s.t. $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

P8 Multiplication is commutative: $\forall a, b \quad a \cdot b = b \cdot a$.

P9 The distributive law: $\forall a, b, c \quad a \cdot (b + c) = a \cdot b + a \cdot c$.

P10 The trichotomy for $\mathbb{P}$: for every a, exactly one of the following holds: $a = 0$, $a \in \mathbb{P}$ or $(-a) \in \mathbb{P}$.

P11 Closure under addition: if a and b are in P, then so is $a + b$.

P12 Closure under multiplication: if a and b are in P, then so is $a \cdot b$.

P13 The thirteenth postulate is the most subtle and interesting of all. It will await a few weeks.

Here are a few corollaries and extra points:

1. Sums such as $a_1 + a_2 + a_3 + \cdots + a_n$ are well defined.
2. The additive identity is unique. (Also multiplicative).
3. Additive inverses are unique. (Also multiplicative).
4. Subtraction can be defined.
5. $a \cdot b = a \cdot c$ iff (if and only if) $a = 0$ or $b = c$.
6. $a \cdot b = 0$ iff $a = 0$ or $b = 0$.
7. $x^2 - 3x + 2 = 0$ iff $x = 1$ or $x = 2$.
8. $a - b = b - a$ iff $a = b$.
9. A "well behaved" order relation can be defined (i.e., the Boolean operations $<$, $\leq$, $>$ and $<$ can be defined and they have all the expected properties).
10. The "absolute value" function $a \mapsto |a|$ can be defined and for all numbers a and b we have

$$|a + b| \leq |a| + |b|.$$