Do not turn this page until instructed.

Math 1100 Core Algebra I

Final Examination

University of Toronto, December 14, 2010

Solve the 5 of the 6 problems on the other side of this page.
Each problem is worth 20 points.
You have three hours to write this test.

Notes.

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.

Good Luck!

Solve 5 of the following 6 problems. Each problem is worth 20 points. You have three hours. **Neatness counts! Language counts!**

Problem 1. Let G be a group.

1. Let H_1 be a finite-index subgroup of G. Prove that there is a normal subgroup N of G, contained in H_1, so that $(G : N)$ is also finite.
2. Let G be a group and H_1 and H_2 be finite-index subgroups of G. Show that $H_1 \cap H_2$ is also of finite index in G.

Hint. Let $(G : H_1) = n$ and find a morphism $G \to S_n$ whose kernel is contained in H_1.

Problem 2. Find all groups of order 1001.
Reference material. $1001 = 7 \cdot 11 \cdot 13 = 7 \cdot 143 = 11 \cdot 91 = 13 \cdot 77$.

Problem 3. Show that in a finite commutative ring, every prime ideal is maximal.

Problem 4. Define a "Principal Ideal Domain (PID)" and a "Unique Factorization Domain (UFD)" and show that every PID is a UFD.

Problem 5.

1. Let A be a 2×2 matrix with entries in a PID R. Show that A can be reduced using invertible row and column operations to a matrix of the form $\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, where $a_1 \mid a_2$.
2. In a paragraph or two, explain why a generalization of the above statement was relevant to our class.

Problem 6. Let a and b be non-zero elements of a PID R, let $g = \gcd(a, b)$ be the greatest common divisor of a and b, and let $m = \operatorname{lcm}(a, b) = ab/g$ be their least common multiple. Without using the structure theorem for finitely generated modules, prove that

$$\frac{R}{\langle a \rangle} \otimes \frac{R}{\langle b \rangle} \cong \frac{R}{\langle g \rangle}, \qquad \text{and} \qquad \frac{R}{\langle a \rangle} \oplus \frac{R}{\langle b \rangle} \cong \frac{R}{\langle g \rangle} \oplus \frac{R}{\langle m \rangle}.$$

Tip. For the second part, you may wish to start with the case $g = 1$ and you'll get partial credit if you stop there.

Good Luck!