

FINITE TYPE INVARIANTS OF W-KNOTTED OBJECTS II: FROM ASSOCIATORS TO SOLUTIONS OF THE KASHIWARA-VERGNE PROBLEM

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ABSTRACT. In this paper we utilize a certain “double tree construction” to show that every “expansion”, namely “universal finite type invariant (UFTI)” of parenthesized braids extends uniquely first to an expansion/UFTI of knotted trivalent graphs (a well known result), and then on to an expansion/UFTI of w-knotted objects, namely to knottings of “2-dimensional foams” and various associated objects in four-dimension space.

In algebraic language, an expansion for parenthesized braids is the same as a “Drinfel’d associator” Φ , and an expansion for w-knotted objects is the same as a solution V of the Kashiwara-Vergne problem [KV] as reformulated by Alekseev and Torossian [AT]. Hence our result amounts to a topological re-interpretation of the result of Alekseev-Enriquez-Torossian [AET] that “there is a formula for V in terms of Φ ”, along with an independent topological proof that the said formula indeed works — that the equations satisfied by V follow from the equations satisfied by Φ .

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1. INTRODUCTION

1.1. **Topology.** We begin by describing a chain of maps from “parenthesized braids” to “(signed) knotted trivalent graphs” to “w-tangled foams”:

$$uPB \xrightarrow{\text{cl}} sKTG \xrightarrow{a} \widetilde{wTF}.$$

Let us first briefly describe each of these spaces and maps.

Parenthesized braids are braids whose ends are ordered along two lines, the “bottom and the ”top“, along with parenthetizations of the endpoints on the bottom and

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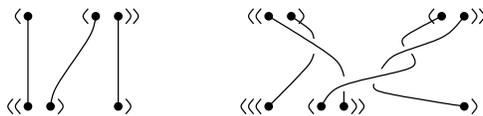


Figure 1. Two examples of parenthesized braids. Note that by convention the parenthetization can be read from the distance scales between the endpoints of the braid, and so we are going to omit the parentheses in the future.

on the top. Two examples are shown in Figure I. Parenthesized braids form a category whose objects are parenthetizations, morphisms are the parenthesized braids themselves, and multiplication is given by stacking. A detailed introduction to parenthesized braids is in [BN1].

2. A BRIEF REVIEW OF uPB , $uKTG$, AND wTF

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Everything below is to be blanked out before the completion of this paper.

RECYCLING

To Do

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