MAT 347
Cyclic groups
September 26, 2018

Definition: A group is *cyclic* if it has a generating set with a single element. In other words, a group G is cyclic if there exists $a \in G$ such that

$$G := \{a^n \mid n \in \mathbb{Z}\}.$$

When this happens, we write $G = \langle a \rangle$.

1. If G is a cyclic group generated by a, what is the relation between $|G|$ and $|a|$? Remember that $|G|$ is the order of G, namely its cardinality. On the other hand, $|a|$ is the order of the element a, which has a different definition.

2. True of False? A group G is cyclic if and only if it contains an element whose order equals $|G|$.

3. Prove that two cyclic groups are isomorphic if and only if they have the same order.

Because of Problem 3, given any positive integer n, we define C_n to be *the* cyclic group of order n. We normally use a multiplicative notation for it. A presentation of C_n would be

$$C_n := \langle a \mid a^n = 1 \rangle.$$

Notice that $C_n \cong \mathbb{Z}/n\mathbb{Z}$ (we still use additive notation for the latter!).

On the other hand, we normally think of $(\mathbb{Z}, +)$ as *the* cyclic group of infinite order, with additive notation.

4. Let G be a cyclic group generated by a. What are all the generators of G? (Here I am asking, which other elements of G generate G?) How many of them are there? You may want to think of the finite and infinite cases separately. If you do not know how to start, consider the specific cases C_6 and C_{12} first.

5. [Do not attempt this question before you have solved the previous four.]

   What can you say about all the subgroups of C_n? This is a long, and somewhat vague question. Here are some ways to make it concrete:

   - Is every subgroup of C_n cyclic?
   - For every $d|n$, how many subgroups of order d does C_n have?
   - For each subgroup of C_n, what are all its generators? How many are there?
   - Which subgroups are contained in each other?

Again, if you do not know what to do, study C_6 and C_{12} first, then make a conjecture, and then try to prove it.

6. Investigate the same questions as in Problem 5, but this time for the infinite cyclic group.