Math 470-1, Fall 2009

Graduate Algebra
Homework 1

1. Let k be a field and let B be the subgroup of $\mathrm{GL}_n(k)$ consisting of upper triangular matrices. Show that B is solvable. [Hint: it's easiest to find an abelian series; but commutators may help to arrive at guessing one.]

2. Let U be the subgroup of $\mathrm{GL}_3(\mathbb{F}_p)$ consisting of upper triangular matrices with diagonal elements equal to 1 (p is any prime). Note that $\#U = p^3$. Find a subgroup H and a normal subgroup N such that $U = N \ltimes H$. Find $H \to \mathrm{Aut}(N)$. [Note: there's in fact precisely one more non-abelian group of order p^3, up to isomorphism.]

3. Consider $G = \langle a, b | a^2 = b^2 = (ab)^2 = 1 \rangle$.

   (a) Use this presentation to find an automorphism of G of order 3.

   (b) What is this group?

4. Consider $G = \langle s, t | s^2 = t^3 = (st)^3 = 1 \rangle$.

   (a) Show that G has at most 12 elements. [Hint: show that each element is represented by a word of length at most 3.]

   (b) Using (a), conclude that $G \xrightarrow{\sim} A_4$.

5. Lang I.14 (p.76), I.28, I.52

6. Suppose that a group G is generated by elements x, y and suppose that $[x, y] = y$. Show that G is solvable. [Hint: $[a, bc] = [a, b] \cdot {}^b[a, c]$, where ${}^bz = bzb^{-1}$.] Show that the order of y is finite and odd. For any odd $d \in \mathbb{N}$, construct an example of such a group with y of order d. [Hint: use a semidirect product.]

   Correction: You need to assume that x has finite order in the second part.

7. Recall that in a free product $\underset{\alpha \in I}{*} G_\alpha$ (I some set), every element can be uniquely expressed as reduced word $g_1 g_2 \dots g_n$, where $g_i \in G_{\alpha_i} - \{1\}$ with $\alpha_i \in I$ and $\alpha_i \neq \alpha_{i+1}$ for all i. We say that such a reduced word is *cyclically reduced* if moreover $\alpha_1 \neq \alpha_n$.

   (a) Show that every element of $\underset{\alpha \in I}{*} G_\alpha$ is conjugate either to a cyclically reduced word or to an element of G_α for some α.

   (b) Deduce that any element of finite order in $\underset{\alpha \in I}{*} G_\alpha$ is conjugate to an element of G_α for some α.

8. For each set S, let $S \to F(S)$ denote a free group on S.

(a) Show that there is a functor from sets to groups that sends S to $F(S)$.

(b) Show that there is a bijection

$$\mathrm{Hom}(F(S), G) \xrightarrow{\sim} \mathrm{Map}(S, G);$$

let's denote it by $\alpha_{S,G}$. Show that this bijection is *natural*: for any set map $S' \to S$ you should get a square-shaped diagram involving $\alpha_{S,G}$ and $\alpha_{S',G}$ and for any group homomorphism $G \to G'$ you should get a square-shaped diagram involving $\alpha_{S,G}$ and $\alpha_{S,G'}$. [Or you get one diagram involving both the map and the homomorphism.] Show that these diagrams commute.